Haskell

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This chapter describes how to install the programs you'll need to start coding in Haskell.

2.1 Installing Haskell

Haskell is a *programming language*, i.e. a language in which humans can express how computers should behave. It's like writing a cooking recipe: you write the recipe and the computer executes it.

To use Haskell programs, you need a special program called a Haskell *compiler*. A compiler takes code written in Haskell and translates it into *machine code*, a more elementary language that the computer understands. Using the cooking analogy, you write a recipe (your Haskell program) and a cook (a compiler program) does the work of putting together actual ingredients into an edible dish (an executable file). Of course, you can't easily get the recipe from a final dish (and you can't get the Haskell program code from executable after it's compiled).

To get started, see haskell.org/downloads[1] for the latest instructions including the "Glasgow Haskell Compiler" (GHC) and everything else you need.

To just test some Haskell basics without downloading and installing, the Haskell.org home page[2] includes a simplified interpreter right on the website. The instructions here in the Wikibook assume the full GHC install, but some of the basics can work in the website version.

Note:
UNIX users:
If you are a person who prefers to compile from source: This might be a bad idea with GHC, especially if it's the first time you install it. GHC is itself mostly written in Haskell, so trying to bootstrap it by hand from source is very tricky. Besides, the build takes a *very* long time and consumes a lot of disk space. If you are sure that you want to build GHC from the source, see Building and Porting GHC at the GHC homepage[a].

[a] http://hackage.haskell.org/trac/ghc/wiki/Building

[1] https://www.haskell.org/downloads

[2] https://www.haskell.org/

2.2 First code

After installation, we will do our first Haskell coding with the program called **GHCi** (the ’i’ stands for ’interactive’). Depending on your operating system, perform the following steps:

- On Windows: Click Start, then Run, then type ’cmd’ and hit Enter, then type `ghci` and hit Enter once more.
- On MacOS: Open the application ”Terminal” found in the ”Applications/Utilities” folder, type the letters `ghci` into the window that appears, and hit the Enter key.
- On Linux: Open a terminal and run `ghci`.

You should get output that looks something like the following:

```
GHCi, version 7.10.1: http://www.haskell.org/ghc/  :? for help
Loading package ghc-prim ... linking ... done.
Loading package integer-gmp ... linking ... done.
Loading package base ... linking ... done.
Prelude>
```

The first bit is GHCi’s version. It then informs you that it’s loading the base package, so you’ll have access to most of the built-in functions and modules that come with GHC. Finally, the `Prelude`> bit is known as the *prompt*. This is where you enter commands, and GHCi will respond with their results.

Now let’s try some basic arithmetic:

```
Prelude> 2 + 2
4
Prelude> 5 + 4 * 3
17
Prelude> 2 ^ 5
32
```

These operators match most other programming languages: `+` is addition, `*` is multiplication, and `^` is exponentiation (raising to the power of, or a^b). As shown in the second example, Haskell follows standard order of math operations (e.g. multiplication before addition).

Now you know how to use Haskell as a calculator. Actually, Haskell is *always* a calculator — just a really powerful one, able to deal not only with numbers but also with other objects like characters, lists, functions, trees, and even other programs (if you aren’t familiar with these terms yet, don’t worry).

GHCi is a powerful development environment. As we progress, we will learn how to load files with source code into GHCi and evaluate different parts of them.

Assuming you’re clear on everything so far (if not, use the talk page and help us improve this Wikibook!), then you are ready for next chapter where we will introduce some of the basic concepts of Haskell and make our first Haskell functions.

3 Variables and functions

All the examples in this chapter can be saved into a Haskell source file and then evaluated by loading that file into GHC. Do not include the "Prelude>" prompts part of any example. When that prompt is shown, it means you can type the following code into an environment like GHCi. Otherwise, you should put the code in a file and run it.

3.1 Variables

In the last chapter, we used GHCi as a calculator. Of course, that's only practical for short calculations. For longer calculations and for writing Haskell programs, we want to keep track of intermediate results.

We can store intermediate results by assigning them names. These names are called *variables*. When a program runs, each variable is substituted for the *value* to which it refers. For instance, consider the following calculation

```
Prelude> 3.141592653 * 5^2
78.539816325
```

That is the approximate area of a circle with radius `5`, according to the formula $A = \pi r^2$. Of course, it is cumbersome to type in the digits of $\pi \approx 3.141592653$, or even to remember more than the first few. Programming helps us avoid mindless repetition and rote memorization by delegating these tasks to a machine. That way, our minds stay free to deal with more interesting ideas. For the present case, Haskell already includes a variable named `pi` that stores over a dozen digits of π for us. This allows for not just clearer code, but also greater precision.

```
Prelude> pi
3.141592653589793
Prelude> pi * 5^2
78.53981633974483
```

Note that the variable `pi` and its value, `3.141592653589793`, can be used interchangeably in calculations.

3.2 Haskell source files

Beyond momentary operations in GHCi, you will save your code in Haskell source files (basically plain text) with the extension `.hs`. Work with these files using a text editor appropriate for coding (see the Wikipedia article on text editors[1]). Proper source code editors will provide *syntax highlighting*, which colors the code in relevant ways to make reading and understanding easier. Vim and Emacs are popular choices among Haskell programmers.

To keep things tidy, create a directory (i.e. a folder) in your computer to save the Haskell files you will create while doing the exercises in this book. Call the directory something like `HaskellWikibook`. Then, create a new file in that directory called `Varfun.hs` with the following code:

```
r = 5.0
```

That code defines the variable `r` as the value `5.0`.

Note: make sure that there are no spaces at the beginning of the line because Haskell is sensitive to whitespace.

Next, with your terminal at the `HaskellWikibook` directory, start GHCi and load the `Varfun.hs` file using the `:load` command:

```
Prelude> :load Varfun.hs
[1 of 1] Compiling Main             ( Varfun.hs, interpreted )
Ok, modules loaded: Main.
```

Note that `:load` can be abbreviated as `:l` (as in `:l Varfun.hs`).

If GHCi gives an error like `Could not find module 'Varfun.hs'`, you probably running GHCi in the wrong directory or saved your file in the wrong directory. You can use the `:cd` command to change directories within GHCi (for instance, `:cd HaskellWikibook`).

With the file loaded, GHCi's prompt changes from ”Prelude” to ”*Main”. You can now use the newly defined variable `r` in your calculations.

```
• Main> r 5.0
• Main> pi * r^2 78.53981633974483
```

So, we calculated the area of a circle with radius of 5.0 using the well-known formula πr^2. This worked because we defined `r` in our Varfun.hs file and `pi` comes from the standard Haskell libraries.

Next, we'll make the area formula easier to quickly access by defining a variable name for it. Change the contents of the source file to:

```
r = 5.0
area = pi * r ^ 2
```

Save the file. Then, assuming you kept GHCi running with the file still loaded, type `:reload` (or abbreviate version `:r`).

[1] https://en.wikipedia.org/wiki/text%20editor

```
• Main> :reload Compiling Main ( Varfun.hs, interpreted ) Ok, modules loaded: Main.
• Main>
```

Now we have two variables `r` and `area`.

```
• Main> area 78.53981633974483
• Main> area / r 15.707963267948966
```

Note:

Note: The `let` *keyword* (a word with a special meaning) lets us define variables directly at the GHCi prompt without a source file. This looks like:

```
Prelude> let area = pi * 5 ^ 2
```

Although sometimes convenient, assigning variables entirely in GHCi this way is impractical for any complex tasks. We will usually want to use saved source files.

3.3 Comments

Besides the working code itself, source files may contain text *comments*. In Haskell there are two types of comment. The first starts with `--` and continues until the end of the line:

```
x = 5     -- x is 5.
y = 6     -- y is 6.
-- z = 7  -- z is not defined.
```

In this case, `x` and `y` are defined in actual Haskell code, but `z` is not.

The second type of comment is denoted by an enclosing `{- ... -}` and can span multiple lines:

```
answer = 2 * {-
  block comment, crossing lines and...
  -} 3 {- inline comment. -} * 7
```

We use comments for explaining parts of a program or making other notes in context. Beware of comment overuse as too many comments can make programs harder to read. Also, we must carefully update comments whenever we change the corresponding code. Outdated comments can cause significant confusion.

3.4 Variables in imperative languages

Readers familiar with imperative programming will notice that variables in Haskell seem quite different from variables in languages like C. If you have no programming experience, you could skip this section, but it will help you understand the general situation when encountering the many cases (most Haskell textbooks, for example) where people discuss Haskell in reference to other programming languages.

Imperative programming treats variables as changeable locations in a computer's memory. That approach connects to the basic operating principles of computers. Imperative programs explicitly tell the computer what to do. Higher-level imperative languages are quite removed from direct computer assembly code instructions, but they retain the same step-by-step way of thinking. In contrast, *functional programming* offers a way to think in higher-level mathematical terms, defining how variables relate to one another, leaving the compiler to translate these to the step-by-step instructions that the computer can process.

Let's look at an example. The following code does not work in Haskell:

```
r = 5
r = 2
```

An imperative programmer may read this as first setting `r = 5` and then changing it to `r = 2`. In Haskell, however, the compiler will respond to the code above with an error: "multiple declarations of `r`". Within a given scope, a variable in Haskell gets defined only once and cannot change.

The variables in Haskell seem almost *invariable*, but they work like variables in mathematics. In a math classroom, you never see a variable change its value within a single problem.

In precise terms, Haskell variables are *immutable*. They vary only based on the data we enter into a program. We can't define `r` two ways in the same code, but we could change the value by changing the file. Let's update our code from above:

```
r = 2.0
area = pi * r ^ 2
```

Of course, that works just fine. We can change `r` in the one place where it is defined, and that will automatically update the value of all the rest of the code that uses the `r` variable.

Real-world Haskell programs work by leaving *some* variables unspecified in the code. The values then get defined when the program gets data from an external file, a database, or user input. For now, however, we will stick to defining variables internally. We will cover interaction with external data in later chapters.

Here's one more example of a major difference from imperative languages:

```
r = r + 1
```

Instead of "incrementing the variable `r`" (i.e. updating the value in memory), this Haskell code is a recursive definition of `r` (i.e. defining it in terms of itself). We will explain recursion[2] in detail later on. For this specific case, if `r` had been defined with any value beforehand, then `r = r + 1`in Haskell would bring an error message. `r = r + 1`is akin to saying, in a mathematical context, that $5 = 5 + 1$, which is plainly wrong.

Because their values do not change within a program, variables can be defined in any order. For example, the following fragments of code do exactly the same thing:

[2] Chapter 12 on page 69

```
y = x * 2
x = 3
```

```
x = 3
y = x * 2
```

In Haskell, there is no notion of ”x being declared before y” or the other way around. Of course, using y will still require a value for x, but this is unimportant until you need a specific numeric value.

3.5 Functions

Changing our program every time we want to calculate the area of new circle is both tedious and limited to one circle at a time. We could calculate two circles by duplicating all the code using new variables `r2` and `area2` for the second circle:[3]

```
r  = 5
area  = pi * r ^ 2
r2 = 3
area2 = pi * r2 ^ 2
```

Of course, to eliminate this mindless repetition, we would prefer to have simply one *function* for area and then apply it to different radii.

A *function* takes an *argument* value (or *parameter*) and gives a result value (essentially the same as in mathematical functions). Defining functions in Haskell is like defining a variable, except that we take note of the function argument that we put on the left hand side. For instance, the following defines a function `area` which depends on an argument named `r`:

```
area r = pi * r ^ 2
```

Look closely at the syntax: the function name comes first (`area` in our example), followed by a space and then the argument (`r` in the example). Following the `=` sign, the *function definition* is a formula that uses the argument in context with other already defined terms.

Now, we can plug in different values for the argument in a *call* to the function. Save the code above in a file, load it into GHCi, and try the following:

```
• Main> area 5 78.53981633974483
• Main> area 3 28.274333882308138
• Main> area 17 907.9202768874502
```

Thus, we can *call* this function with different radii to calculate the area of any circle.

Our function here is defined mathematically as

$$A(r) = \pi \cdot r^2$$

[3] As this example shows, the names of variables may contain numbers as well as letters. Variables in Haskell *must* begin with a lowercase letter but may then have any string consisting of letter, numbers, underscore (_) or tick (').

In mathematics, the parameter is enclosed between parentheses, as in $A(5) = 78.54$ or $A(3) = 28.27$. Haskell code will also work with parentheses, but we omit them as a convention. Haskell uses functions all the time, and whenever possible we want to minimize extra symbols.

We still use parentheses for grouping *expressions* (any code that gives a value) that must be evaluated together. Note how the following expressions are parsed differently:

```
5 * 3 + 2       -- 15 + 2 = 17 (multiplication is done before addition)
5 * (3 + 2)     -- 5 * 5 = 25 (thanks to the parentheses)
area 5 * 3      -- (area 5) * 3
area (5 * 3)    -- area 15
```

Note that Haskell functions take *precedence* over all other operators such as `+` and `*`, in the same way that, for instance, multiplication is done before addition in mathematics.

3.5.1 Evaluation

What exactly happens when you enter an expression into GHCi? After you press the enter key, GHCi will *evaluate* the expression you have given. That means it will replace each function with its definition and calculate the results until a single value remains. For example, the evaluation of `area 5` proceeds as follows:

```
   area 5
=>    { replace the left-hand side  area r = ...  by the right-hand side  ... =
pi * r^2 }
   pi * 5 ^ 2
=>    { replace  pi  by its numerical value }
   3.141592653589793 * 5 ^ 2
=>    { apply exponentiation (^) }
   3.141592653589793 * 25
=>    { apply multiplication (*) }
   78.53981633974483
```

As this shows, to *apply* or *call* a function means to replace the left-hand side of its definition by its right-hand side. When using GHCi, the results of a function call will then show on the screen.

Some more functions:

```
double x    = 2 * x
quadruple x = double (double x)
square x    = x * x
half   x    = x / 2
```

Exercises:

- Explain how GHCi evaluates `quadruple 5`.
- Define a function that subtracts 12 from half its argument.

3.5.2 Multiple parameters

Functions can also take more than one argument. For example, a function for calculating the area of a rectangle given its length and width:

```
areaRect l w = l * w
```

```
• Main> areaRect 5 10 50
```

Another example that calculates the area of a triangle $\left(A = \frac{bh}{2}\right)$:

```
areaTriangle b h = (b * h) / 2
```

```
• Main> areaTriangle 3 9 13.5
```

As you can see, multiple arguments are separated by spaces. That's also why you sometimes have to use parentheses to group expressions. For instance, to quadruple a value `x`, you can't write

```
quadruple x = double double x     -- error
```

That would apply a function named `double` to the two arguments `double` and `x`. Note that *functions can be arguments to other functions* (you will see why later). To make this example work, we need to put parentheses around the argument:

```
quadruple x = double (double x)
```

Arguments are always passed in the order given. For example:

```
minus x y = x - y
```

```
• Main> minus 10 5 5
• Main> minus 5 10 -5
```

Here, `minus 10 5` evaluates to `10 - 5`, but `minus 5 10` evaluates to `5 - 10` because the order changes.

Exercises:

- Write a function to calculate the volume of a box.
- Approximately how many stones are the famous pyramids at Giza made up of? Hint: you will need estimates for the volume of the pyramids and the volume of each block.

3.5.3 On combining functions

Of course, you can use functions that you have already defined to define new functions, just like you can use the predefined functions like addition `(+)` or multiplication `(*)` (operators are defined as functions in Haskell). For example, to calculate the area of a square, we can reuse our function that calculates the area of a rectangle:

```
areaRect l w = l * w
areaSquare s = areaRect s s
```

- Main> areaSquare 5 25

After all, a square is just a rectangle with equal sides.

Exercises:

- Write a function to calculate the volume of a cylinder. The volume of a cylinder is the area of the base, which is a circle (you already programmed this function in this chapter, so reuse it) multiplied by the height.

3.6 Local definitions

3.6.1 `where` clauses

When defining a function, we sometimes want to define intermediate results that are *local* to the function. For instance, consider Heron's formula[4] $A = \sqrt{s(s-a)(s-b)(s-c)}$ for calculating the area of a triangle with sides `a`, `b`, and `c`:

```
heron a b c = sqrt (s * (s - a) * (s - b) * (s - c))
    where
    s = (a + b + c) / 2
```

The variable `s` is half the perimeter of the triangle and it would be tedious to write it out four times in the argument of the square root function `sqrt`.

Simply writing the definitions in sequence does not work...

```
heron a b c = sqrt (s * (s - a) * (s - b) * (s - c))
s = (a + b + c) / 2                                  -- a, b, and c are not
 defined here
```

... because the variables `a`, `b`, `c` are only available in the right-hand side of the function `heron`, but the definition of `s` as written here is not part of the right-hand side of `heron`. To make it part of the right-hand side, we use the `where` keyword.

Note that both the `where` and the local definitions are *indented* by 4 spaces, to distinguish them from subsequent definitions. Here is another example that shows a mix of local and top-level definitions:

```
areaTriangleTrig  a b c = c * height / 2   -- use trigonometry
    where
    cosa   = (b ^ 2 + c ^ 2 - a ^ 2) / (2 * b * c)
```

[4] https://en.wikipedia.org/wiki/Heron%27s%20formula

```
    sina   = sqrt (1 - cosa ^ 2)
    height = b * sina
areaTriangleHeron a b c = result           -- use Heron's formula
    where
    result = sqrt (s * (s - a) * (s - b) * (s - c))
    s      = (a + b + c) / 2
```

3.6.2 Scope

If you look closely at the previous example, you'll notice that we have used the variable names `a`, `b`, `c` twice, once for each of the two area functions. How does that work?

Consider the following GHCi sequence:

```
Prelude> let r = 0
Prelude> let area r = pi * r ^ 2
Prelude> area 5
78.53981633974483
```

It would have been an unpleasant surprise to return `0` for the area because of the earlier `let r = 0` definition getting in the way. That does not happen because when you defined `r` the second time you are talking about a *different* `r`. This may seem confusing, but consider how many people have the name John, and yet for any context with only one John, we can talk about "John" with no confusion. Programming has a notion similar to context, called *scope*[5].

We will not explain the technicalities behind scope right now. Just keep in mind that the value of a parameter is strictly what you pass in when you call the function, regardless of what the variable was called in the function's definition. That said, appropriately unique names for variables do make the code easier for human readers to understand.

3.7 Summary

1. Variables store values (which can be any arbitrary Haskell expression).
2. Variables do not change within a scope.
3. Functions help you write reusable code.
4. Functions can accept more than one parameter.

We also learned about non-code text comments within a source file.

[5] `https://en.wikipedia.org/wiki/Scope%20%28programming%29`

4 Truth values

4.1 Equality and other comparisons

In the last chapter, we used the equals sign to define variables and functions in Haskell as in the following code:

```
r = 5
```

That means that the evaluation of the program replaces all occurrences of `r` with `5` (within the scope of the definition). Similarly, evaluating the code

```
f x = x + 3
```

replaces all occurrences of `f` followed by a number (`f`'s argument) with that number plus three.

Mathematics also uses the equals sign in an important and subtly different way. For instance, consider this simple problem:

Example: Solve the following equation:
$x+3=5$

Our interest here isn't about representing the value 5 as $x+3$, or vice-versa. Instead, we read the $x+3=5$ equation as a *proposition* that some number x gives 5 as result when added to 3. Solving the equation means finding which, if any, values of x make that proposition true. In this example, elementary algebra tells us that $x=2$ (i.e. 2 is the number that will make the equation true, giving $2+3=5$).

Comparing values to see if they are equal is also useful in programming. In Haskell, such tests look just like an equation. Since the equals sign is already used for defining things, Haskell uses a *double* equals sign, `==` instead. Enter our proposition above in GHCi:

```
Prelude> 2 + 3 == 5
True
```

GHCi returns ”True” because $2+3$ is equal to 5. What if we use an equation that is not true?

```
Prelude> 7 + 3 == 5
False
```

Nice and coherent. Next, we will use our own functions in these tests. Let's try the function `f` we mentioned at the start of the chapter:

```
Prelude> let f x = x + 3
Prelude> f 2 == 5
True
```

This works as expected because `f 2` evaluates to `2 + 3`.

We can also compare two numerical values to see which one is larger. Haskell provides a number of tests including: `<` (less than), `>` (greater than), `<=` (less than or equal to) and `>=` (greater than or equal to). These tests work comparably to `==` (equal to). For example, we could use `<` alongside the `area` function from the previous module to see whether a circle of a certain radius would have an area smaller than some value.

```
Prelude> let area r = pi * r ^ 2
Prelude> area 5 < 50
False
```

4.2 Boolean values

What is actually going on when GHCi determines whether these arithmetical propositions are true or false? Consider a different but related issue. If we enter an arithmetical expression in GHCi the expression gets *evaluated*, and the resulting numerical value is displayed on the screen:

```
Prelude> 2 + 2
4
```

If we replace the arithmetical expression with an equality comparison, something similar seems to happen:

```
Prelude> 2 == 2
True
```

Whereas the ”4” returned earlier is a number which represents some kind of count, quantity, etc., ”True” is a *value* that stands for the truth of a proposition. Such values are called **truth values**, or **boolean values**.[1] Naturally, only two possible boolean values exist: `True` and `False`.

[1] The term is a tribute to the mathematician and philosopher George Boole ^{`https://en.wikipedia.org/wiki/George%20Boole`} .

4.2.1 Introduction to types

`True` and `False` are real values, not just an analogy. Boolean values have the same status as numerical values in Haskell, and you can manipulate them in similar ways. One trivial example:

```
Prelude> True == True
True
Prelude> True == False
False
```

`True` is indeed equal to `True`, and `True` is not equal to `False`. Now: can you answer whether `2` is equal to `True`?

```
Prelude> 2 == True

<interactive>:1:0:
```

```
      No instance for (Num Bool)
        arising from the literal ‘2’ at <interactive>:1:0
      Possible fix: add an instance declaration for (Num Bool)
      In the first argument of ‘(==)’, namely ‘2’
      In the expression: 2 == True
      In an equation for ‘it’: it = 2 == True
```

Error! The question just does not make sense. We cannot compare a number with a non-number or a boolean with a non-boolean. Haskell incorporates that notion, and the ugly error message complains about this. Ignoring much of the clutter, the message says that there was a number (`Num`) on the left side of the `==`, and so some kind of number was expected on the right side; however, a boolean value (`Bool`) is not a number, and so the equality test failed.

So, values have **types**, and these types define limits to what we can or cannot do with the values. `True` and `False` are values of type `Bool`. The `2` is complicated because there are many different types of numbers, so we will defer that explanation until later. Overall, types provide great power because they regulate the behavior of values with rules that *make sense*, making it easier to write programs that work correctly. We will come back to the topic of types many times as they are very important to Haskell.

4.3 Infix operators

An equality test like `2 == 2` is an expression just like `2 + 2`; it evaluates to a value in pretty much the same way. The ugly error message we got on the previous example even says so:

```
      In the expression: 2 == True
```

When we type `2 == 2` in the prompt and GHCi ”answers” `True`, it is simply evaluating an expression. In fact, `==` is itself a function which takes two arguments (which are the left side and the right side of the equality test), but the syntax is notable: Haskell allows two-argument functions to be written as *infix operators* placed *between* their arguments. When the function name uses only non-alphanumeric characters, this infix approach is the common use case. If you wish to use such a function in the ”standard” way (writing the function name before the arguments, as a *prefix operator*) the function name must be enclosed in parentheses. So the following expressions are completely equivalent:

```
Prelude> 4 + 9 == 13
True
Prelude> (==) (4 + 9) 13
True
```

Thus, we see how `(==)` works as a function similarly to `areaRect` from the previous module. The same considerations apply to the other *relational operators* we mentioned (`<`, `>`, `<=`, `>=`) and to the arithmetical operators (`+`, `*`, etc.) – all are functions that take two arguments and are normally written as infix operators.

In general, we can say that tangible things in Haskell are either values or functions.

4.4 Boolean operations

Haskell provides three basic functions for further manipulation of truth values as in logic propositions:

- `(&&)` performs the *and* operation. Given two boolean values, it evaluates to `True` if both the first and the second are `True`, and to `False` otherwise.

```
Prelude> (3 < 8) && (False == False)
True
Prelude> (&&) (6 <= 5) (1 == 1)
False
```

- `(||)` performs the *or* operation. Given two boolean values, it evaluates to `True` if either the first or the second are `True` (or if both are true), and to `False` otherwise.

```
Prelude> (2 + 2 == 5) || (2 > 0)
True
Prelude> (||) (18 == 17) (9 >= 11)
False
```

- `not` performs the negation of a boolean value; that is, it converts `True` to `False` and vice-versa.

```
Prelude> not (5 * 2 == 10)
False
```

Haskell libraries already include the relational operator function `(/=)` for *not equal to*, but we could easily implement it ourselves as:

```
x /= y = not (x == y)
```

Note that we can write operators infix even when defining them. Completely new operators can also be created out of ASCII symbols (which means mostly the common symbols used on a keyboard).

4.5 Guards

Haskell programs often use boolean operators in convenient and abbreviated syntax. When the same logic is written in alternative styles, we call this *syntactic sugar* because it sweetens the code from the human perspective. We'll start with *guards*, a feature that relies on boolean values and allows us to write simple but powerful functions.

Let's implement the absolute value function. The absolute value of a real number is the number with its sign discarded; so if the number is negative (that is, smaller than zero) the sign is inverted; otherwise it remains unchanged. We could write the definition as:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Here, the actual expression to be used for calculating $|x|$ depends on a set of propositions made about x. If $x \geq 0$ is true, then we use the first expression, but if $x < 0$ is the case, then we use the second expression instead. To express this decision process in Haskell using guards, the implementation could look like this:[2]

Example: The absolute value function.

```
absolute x
    | x < 0     = 0 - x
    | otherwise = x
```

Remarkably, the above code is about as readable as the corresponding mathematical definition. Let us dissect the components of the definition:

- We start just like a normal function definition, providing a name for the function, `absolute`, and saying it will take a single argument, which we will name `x`.

[2] This function is already provided by Haskell with the name `abs`, so in a real-world situation you don't need to provide an implementation yourself.

- Instead of just following with the = and the right-hand side of the definition, we enter the two alternatives placed below on separate lines.[3] These alternatives are the *guards* proper. Note that the whitespace (the indentation of the second and third lines) is not just for aesthetic reasons; it is necessary for the code to be parsed correctly.
- Each of the guards begins with a pipe character, `|`. After the pipe, we put an expression which evaluates to a boolean (also called a boolean condition or a *predicate*), which is followed by the rest of the definition. The function only uses the equals sign and the right-hand side from a line if the predicate evaluates to `True`.
- The `otherwise` case is used when none of the preceding predicates evaluate to `True`. In this case, if `x` is not smaller than zero, it must be greater than or equal to zero, so the final predicate could have just as easily been `x >= 0`; but `otherwise` works just as well.

Note:
There is no syntactical magic behind `otherwise`. It is defined alongside the default variables and functions of Haskell as simply

```
otherwise = True
```

This definition makes *otherwise* a catch-all guard. As evaluation of the guard predicates is sequential, the `otherwise` predicate will only be reached if none of the previous cases evaluate to `True` (so make sure you always place *otherwise* as the last guard!). In general, it is a good idea to always provide an `otherwise` guard, because a rather ugly runtime error will be produced if none of the predicates is true for some input.

Note:
You might wonder why we wrote `0 - x` and not simply `-x` to denote the sign inversion. Well, we could have written the first guard as

```
    | x < 0    = -x
```

and that would work, but this way of expressing sign inversion is one of a few ”special cases” in Haskell; the `-` is *not* a function that takes one argument and evaluates to `0 - x`, it's a syntactical abbreviation. While very handy, this shortcut occasionally conflicts with the usage of `(-)` as an actual function (the subtraction operator), which is a potential source of annoyance (for example, try writing three minus negative-four without using any parentheses for grouping). So, we wrote `0 - x` explicitly so that we could point out this issue.

4.5.1 `where` and Guards

`where` clauses work well along with guards. For instance, consider a function which computes the number of (real) solutions for a quadratic equation[4], $ax^2 + bx + c = 0$:

[3] We *could* have joined the lines and written everything in a single line, but it would be less readable.

[4] https://en.wikipedia.org/wiki/Quadratic%20equation

```
numOfRealSolutions a b c
    | disc > 0  = 2
    | disc == 0 = 1
    | otherwise = 0
      where
      disc = b^2 - 4*a*c
```

The `where` definition is within the scope of all of the guards, sparing us from repeating the expression for `disc`.

5 Type basics

In programming, **Types** are used to group similar values into categories. In Haskell, the type system is a powerful way of reducing the number of mistakes in your code.

5.1 Introduction

Programming deals with different sorts of entities. For example, consider adding two numbers together:

$2+3$

What are 2 and 3? Well, they are numbers. What about the plus sign in the middle? That's certainly not a number, but it stands for an operation which we can do with two numbers – namely, addition.

Similarly, consider a program that asks you for your name and then greets you with a ”Hello” message. Neither your name nor the word Hello are numbers. What are they then? We might refer to all words and sentences and so forth as text. It's normal in programming to use a slightly more esoteric word: *String*, which is short for ”string of characters”.

Note:
Haskell has a rule that all type names have to begin with a capital letter. We shall adhere to this convention henceforth.

Databases illustrate clearly the concept of types. For example, say we had a table in a database to store details about a person's contacts; a kind of personal telephone book. The contents might look like this:

First Name	Last Name	Address	Telephone number
Sherlock	Holmes	221B Baker Street London	743756
Bob	Jones	99 Long Road Street Villestown	655523

The fields in each entry contain values. `Sherlock` is a value as is `99 Long Road Street Villestown` as well as `655523`. Let's classify the values in this example in terms of types. ”First Name” and ”Last Name” contain text, so we say that the values are of type String.

At first glance, we might classify address as a String. However, the semantics behind an innocent address are quite complex. Many human conventions dictate how we interpret addresses. For example, if the beginning of the address text contains a number it is likely the number of the house. If not, then it's probably the name of the house – except if it starts with ”PO Box”, in which case it’s just a postal box address and doesn’t indicate where the person lives at all. Each part of the address has its own meaning.

In principle, we can indeed say that addresses are Strings, but that doesn’t capture many important features of addresses. When we describe something as a String, all that we are saying is that it is a sequence of characters (letters, numbers, etc). Recognizing something as a specialized type is far more meaningful. If we know something is an Address, we instantly know much more about the piece of data – for instance, that we can interpret it using the ”human conventions” that give meaning to addresses.

We might also apply this rationale to the telephone numbers. We could specify a TelephoneNumber type. Then, if we were to come across some arbitrary sequence of digits which happened to be of type TelephoneNumber we would have access to a lot more information than if it were just a Number – for instance, we could start looking for things such as area and country codes on the initial digits.

Another reason not to consider the telephone numbers as Numbers is that doing arithmetics with them makes no sense. What is the meaning and expected effect of, say, multiplying a TelephoneNumber by 100? It would not allow calling anyone by phone. Also, each digit comprising a telephone number is important; we cannot accept losing some of them by rounding or even by omitting leading zeros.

5.1.1 Why types are useful

How does it help us program well to describe and categorize things? Once we define a type, we can specify what we can or cannot do with it. That makes it far easier to manage larger programs and avoid errors.

5.2 Using the interactive `:type` command

Let’s explore how types work using GHCi. The type of any expression can be checked with `:type` (or shortened to `:t`) command. Try this on the boolean values from the previous module:

Example:
Exploring the types of boolean values in GHCi

```
Prelude> :type True
True :: Bool
Prelude> :type False
False :: Bool
Prelude> :t (3 < 5)
(3 < 5) :: Bool
```

The symbol `::`, which will appear in a couple other places, can be read as simply ”is of type”, and indicates a *type signature.*

`:type` reveals that truth values in Haskell are of type `Bool`, as illustrated above for the two possible values, `True` and `False`, as well as for a sample expression that will evaluate to one of them. Note that boolean values are not just for value comparisons. `Bool` captures the semantics of a yes/no answer, so it can represent any information of such kind – say, whether a name was found in a spreadsheet, or whether a user has toggled an on/off option.

5.2.1 Characters and strings

Now let's try `:t` on something new. Literal characters are entered by enclosing them with single quotation marks. For instance, this is the single letter H:

Example:
Using the :type command in GHCi on a literal character
Prelude> :t 'H'
'H' :: Char

So, literal character values have type `Char` (short for "character"). Now, single quotation marks only work for individual characters, so if we need to enter longer text – that is, a string of characters – we use *double* quotation marks instead:

Example:
Using the :t command in GHCi on a literal string
Prelude> :t "Hello World"
"Hello World" :: [Char]

Why did we get `Char` again? The difference is the square brackets. `[Char]` means a number of characters chained together, forming a *list* of characters. Haskell considers all Strings to be lists of characters. Lists in general are important entities in Haskell, and we will cover them in more detail in a little while.

Exercises:

1. Try using `:type` on the literal value `"H"` (notice the double quotes). What happens? Why?
2. Try using `:type` on the literal value `'Hello World'` (notice the single quotes). What happens? Why?

Incidentally, Haskell allows for *type synonyms*, which work pretty much like synonyms in human languages (words that mean the same thing – say, 'big' and 'large'). In Haskell, type synonyms are alternative names for types. For instance, `String` is defined as a synonym of `[Char]`, and so we can freely substitute one with the other. Therefore, to say:

```
"Hello World" :: String
```

is also perfectly valid, and in many cases a lot more readable. From here on we'll mostly refer to text values as `String`, rather than `[Char]`.

5.3 Functional types

So far, we have seen how values (strings, booleans, characters, etc.) have types and how these types help us to categorize and describe them. Now, the big twist that makes Haskell's type system truly powerful: *Functions* have types as well.[1] Let's look at some examples to see how that works.

5.3.1 Example: `not`

We can negate boolean values with `not` (e.g. `not True` evaluates to `False` and vice-versa). To figure out the type of a function, we consider two things: the type of values it takes as its input and the type of value it returns. In this example, things are easy. `not` takes a `Bool` (the `Bool` to be negated), and returns a `Bool` (the negated `Bool`). The notation for writing that down is:

Example: Type signature for `not`

```
not :: Bool -> Bool
```

You can read this as "`not` is a function from things of type `Bool` to things of type `Bool`".

Using `:t` on a function will work just as expected:

```
Prelude> :t not
not :: Bool -> Bool
```

The description of a function's type is in terms of the types of argument(s) it takes and the type of value it evaluates to.

5.3.2 Example: `chr` and `ord`

Text presents a problem to computers. At its lowest level, a computer only knows binary 1s and 0s. To represent text, every character is first converted to a number, then that number is converted to binary and stored. That's how a piece of text (which is just a sequence of characters) is encoded into binary. Normally, we're only interested in how to encode characters into their numerical representations, because the computer takes care of the conversion to binary numbers without our intervention.

The easiest way to convert characters to numbers is simply to write all the possible characters down, then number them. For example, we might decide that 'a' corresponds to 1, then 'b' to 2, and so on. This is what something called the ASCII standard is: take 128 commonly-used characters and number them (ASCII doesn't actually start with 'a', but the general idea is the same). Of course, it would be quite a chore to sit down and look

[1] The deeper truth is that functions *are* values, just like all the others.

up a character in a big lookup table every time we wanted to encode it, so we've got two functions that do it for us, `chr` (pronounced 'char') and `ord`[2]:

Example: Type signatures for `chr` and `ord`

```
chr :: Int  -> Char
ord :: Char -> Int
```

We already know what `Char` means. The new type on the signatures above, `Int`, refers to integer numbers, and is one of quite a few different types of numbers.[3] The type signature of `chr` tells us that it takes an argument of type `Int`, an integer number, and evaluates to a result of type Char. The converse is the case with `ord`: It takes things of type Char and returns things of type Int. With the info from the type signatures, it becomes immediately clear which of the functions encodes a character into a numeric code (`ord`) and which does the decoding back to a character (`chr`).

To make things more concrete, here are a few examples. Notice that the two functions aren't available by default; so before trying them in GHCi you need to use the `:module Data.Char` (or `:m Data.Char`) command to load the Data.Char module where they are defined.

Example:
Function calls to <code>chr</code> and <code>ord</code>

```
Prelude> :m Data.Char
Prelude Data.Char> chr 97
'a'
Prelude Data.Char> chr 98
'b'
Prelude Data.Char> ord 'c'
99
```

5.3.3 Functions with more than one argument

What would be the type of a function that takes more than one argument?

Example: A function with more than one argument

```
xor p q = (p || q) && not (p && q)
```

[2] This isn't quite what `chr` and `ord` do, but that description fits our purposes well, and it's close enough.

[3] In fact, it is not even the only type for integers! We will meet its relatives in a short while.

(xor is the exclusive-or function, which evaluates to True if either one or the other argument is True, *but not both*; and False otherwise.)

The general technique for forming the type of a function that accepts more than one argument is simply to write down all the types of the arguments in a row, in order (so in this case p first then q), then link them all with ->. Finally, add the type of the result to the end of the row and stick a final -> in just before it.[4] In this example, we have:

1. Write down the types of the arguments. In this case, the use of (||) and (&&) gives away that p and q have to be of type Bool:

```
Bool                   Bool
^^ p is a Bool         ^^ q is a Bool as well
```

2. Fill in the gaps with ->:
```
Bool -> Bool
```
3. Add in the result type and a final ->. In our case, we're just doing some basic boolean operations so the result remains a Bool.

```
Bool -> Bool -> Bool
                ^^ We're returning a Bool
            ^^ This is the extra -> that got added in
```

The final signature, then, is:

Example: The signature of xor

```
xor :: Bool -> Bool -> Bool
```

5.3.4 Real world example: openWindow

Note:
A library is a collection of common code used by many programs.

As you'll learn in the Haskell in Practice section of the course, one popular group of Haskell libraries are the GUI (**G**raphical **U**ser **I**nterface) ones. These provide functions for dealing with the visual things computer users are familiar with: menus, buttons, application windows, moving the mouse around, etc. One function from one of these libraries is called openWindow, and you can use it to open a new window in your application. For example, say you're writing a word processor, and the user has clicked on the 'Options' button. You need to open a new window which contains all the options that they can change. Let's look at the type signature for this function:[5]

[4] This method might seem just a trivial hack by now, but actually there are very deep reasons behind it, which we'll cover in the chapter on higher-order functions ^{Chapter19 on page 117}.

[5] This has been somewhat simplified to fit our purposes. Don't worry, the essence of the function is there.

Example: `openWindow`

```
openWindow :: WindowTitle -> WindowSize -> Window
```

You don't know these types, but they're quite simple. All three of the types there, `WindowTitle`, `WindowSize` and `Window` are defined by the GUI library that provides `openWindow`. As we saw earlier, the two arrows mean that the first two types are the types of the parameters, and the last is the type of the result. `WindowTitle` holds the title of the window (which typically appears in a title bar at the very top of the window), and `WindowSize` specifies how big the window should be. The function then returns a value of type `Window` which represents the actual window.

So, even if you have never seen a function before or don't know how it actually works, a type signature can give you a general idea of what the function does. Make a habit of testing every new function you meet with `:t`. If you start doing that now, you'll not only learn about the standard library Haskell functions but also develop a useful kind of intuition about functions in Haskell.

Exercises:
What are the types of the following functions? For any functions involving numbers, you can just pretend the numbers are Ints.

1. The `negate` function, which takes an Int and returns that Int with its sign swapped. For example, `negate 4 = -4`, and `negate (-2) = 2`
2. The `(||)` function, pronounced 'or', that takes two Bools and returns a third Bool which is True if either of the arguments were, and False otherwise.
3. A `monthLength` function which takes a Bool which is True if we are considering a leap year and False otherwise, and an Int which is the number of a month; and returns another Int which is the number of days in that month.
4. `f x y = not x && y`
5. `g x = (2*x - 1)^2`

5.4 Type signatures in code

We have explored the basic theory behind types and how they apply to Haskell. Now, we will see how type signatures are used for annotating functions in source files. Consider the `xor` function from an earlier example:

Example: A function with its signature

```
xor :: Bool -> Bool -> Bool
xor p q = (p || q) && not (p && q)
```

That is all we have to do. For maximum clarity, type signatures go above the corresponding function definition.

The signatures we add in this way serve a dual role: they clarify the type of the functions both to human readers and to the compiler/interpreter.

5.4.1 Type inference

If type signatures tell the interpreter (or compiler) about the function type, how did we write our earliest Haskell code without type signatures? Well, when you don't tell Haskell the types of your functions and variables it figures them out through a process called *type inference*. In essence, the compiler starts with the types of things it knows and then works out the types of the rest of the values. Consider a general example:

Example: Simple type inference

```
-- We're deliberately not providing a type signature for this function
isL c = c == 'l'
```

`isL` is a function that takes an argument `c` and returns the result of evaluating `c == 'l'`. Without a type signature, the type of `c` and the type of the result are not specified. In the expression `c == 'l'`, however, the compiler knows that `'l'` is a `Char`. Since `c` and `'l'` are being compared with equality with `(==)` and both arguments of `(==)` must have the same type,[6] it follows that `c` must be a `Char`. Finally, since `isL c` is the result of `(==)` it must be a `Bool`. And thus we have a signature for the function:

Example: `isL` with a type

```
isL :: Char -> Bool
isL c = c == 'l'
```

Indeed, if you leave out the type signature, the Haskell compiler will discover it through this process. You can verify that by using `:t` on `isL` with or without a signature.

So why write type signatures if they will be inferred anyway? In some cases, the compiler lacks information to infer the type, and so the signature becomes obligatory. In some other cases, we can use a type signature to influence to a certain extent the final type of a function or value. These cases needn't concern us for now, but we have a few other reasons to include type signatures:

- **Documentation**: type signatures make your code easier to read. With most functions, the name of the function along with the type of the function is sufficient to guess what

[6] As discussed in Truth values ^{Chapter4 on page 17}. That fact is actually stated by the type signature of `(==)` – if you are curious you can check it, although you will have to wait a little bit more for a full explanation of the notation used in that.

the function does. Of course, commenting your code helps, but having the types clearly stated helps too.

- **Debugging**: when you annotate a function with a type signature and then make a typo in the body of the function which changes the type of a variable, the compiler will tell you, *at compile-time*, that your function is wrong. Leaving off the type signature might allow your erroneous function to compile, and the compiler would assign it the wrong type. You wouldn't know until you ran your program that you made this mistake.

5.4.2 Types and readability

A somewhat more realistic example will help us understand better how signatures can help documentation. The piece of code quoted below is a tiny *module* (modules are the typical way of preparing a library), and this way of organizing code is like that in the libraries bundled with GHC.

Note:
Do not go crazy trying to understand how the functions here actually work; that is beside the point as we still have not covered many of the features being used. Just keep reading and play along.

Example: Module with type signatures

```haskell
module StringManip where

import Data.Char

uppercase, lowercase :: String -> String
uppercase = map toUpper
lowercase = map toLower

capitalize :: String -> String
capitalize x =
  let capWord []     = []
      capWord (x:xs) = toUpper x : xs
  in unwords (map capWord (words x))
```

This tiny library provides three string manipulation functions. `uppercase` converts a string to upper case, `lowercase` to lower case, and `capitalize` capitalizes the first letter of every word. Each of these functions takes a `String` as argument and evaluates to another `String`. Even if we do not understand how these functions work, looking at the type signatures allows us to immediately know the types of the arguments and return values. Paired with sensible function names, we have enough information to figure out how we can use the functions.

Note that when functions have the same type we have the option of writing just one signature for all of them, by separating their names with commas, as above with `uppercase` and `lowercase`.

5.4.3 Types prevent errors

The role of types in preventing errors is central to typed languages. When passing expressions around you have to make sure the types match up like they did here. If they don't, you'll get *type errors* when you try to compile; your program won't pass the *typecheck*. This helps reduce bugs in your programs. To take a very trivial example:

Example: A non-typechecking program

```
"hello" + " world"     -- type error
```

That line will cause a program to fail when compiling. You can't add two strings together. In all likelihood, the programmer intended to use the similar-looking concatenation operator, which can be used to join two strings together into a single one:

Example: Our erroneous program, fixed

```
"hello" ++ " world"    -- "hello world"
```

An easy typo to make, but Haskell catches the error when you tried to compile. You don't have to wait until you run the program for the bug to become apparent.

Updating a program commonly involves changes to types. If a change is unintended, or has unforeseen consequences, then it will show up when compiling. Haskell programmers often remark that once they have fixed all the type errors, and their programs compile, that they tend to ”just work”. The behavior may not always match the intention, but the program won't crash. Haskell has far fewer *run-time errors* (where your program goes wrong when you run it rather than when you compile) than other languages.

6 Lists and tuples

7 Type basics II

In this chapter, we will show how numerical types are handled in Haskell and introduce some important features of the type system. Before diving into the text, though, pause for a moment and consider the following question: what should be the type of the function `(+)`?[1]

7.1 The `Num` class

Mathematics puts restrictions on the kind of numbers we can add together. $2+3$ (two natural numbers), $(-7)+5.12$ (a negative integer and a rational number), $\frac{1}{7}+\pi$ (a rational and an irrational). All of these are valid. In fact any two real numbers can be added together. In order to capture such generality in the simplest way possible we need a general `Number` type in Haskell, so that the signature of `(+)` would be simply

```
(+) :: Number -> Number -> Number
```

However, that design fits poorly with the way computers perform arithmetic. While computers can handle integers as a sequence of bits in memory, that approach does not work for real numbers,[2] thus making it necessary for a less than perfect encoding for them: floating point numbers[3]. While floating point provides a reasonable way to deal with real numbers in general, it has some inconveniences (most notably, loss of precision) which makes using the simpler encoding worthwhile for integer values. So, we have at least two different ways of storing numbers: one for integers and another for general real numbers. Each approach should correspond to different Haskell types. Furthermore, computers are only able to perform operations like `(+)` on a pair of numbers if they are in the same format.

So much for having a universal `Number` type – it seems that we can't even have `(+)` mix integers and floating-point numbers. However, Haskell *can* at least use the same `(+)` function with either integers or floating point numbers. Check this yourself in GHCi:

```
Prelude>3 + 4
7
Prelude>4.34 + 3.12
7.46
```

[1] If you followed our recommendations in ”Type basics”, chances are you have already seen the rather exotic answer by testing with `:t`... if that is the case, consider the following analysis as a path to understanding the meaning of that signature.

[2] Among other issues, between any two real numbers there are uncountably many real numbers – and that fact can't be directly mapped into a representation in memory no matter what we do.

[3] `https://en.wikipedia.org/wiki/Floating%20point`

When discussing lists and tuples, we saw that functions can accept arguments of different types if they are made *polymorphic*. In that spirit, here's a possible type signature for `(+)` that would account for the facts above:

```
(+) :: a -> a -> a
```

With that type signature, `(+)` would take two arguments of the same type `a` (which could be integers or floating-point numbers) and evaluate to a result of type `a` (as long as both arguments are the same type). But this type signature indicates *any* type at all, and we know that we can't use `(+)` with two `Bool` values, or two `Char` values. What would adding two letters or two truth-values mean? So, the *actual* type signature of `(+)` uses a language feature that allows us to express the semantic restriction that `a` can be any type *as long as it is a number type*:

```
(+) :: (Num a) => a -> a -> a
```

`Num` is a **typeclass** — a group of types which includes all types which are regarded as numbers.[4] The `(Num a) =>` part of the signature restricts `a` to number types – or, in Haskell terminology, *instances* of `Num`.

7.2 Numeric types

So, which are the *actual* number types (that is, the instances of `Num` that the `a` in the signature may stand for)? The most important numeric types are `Int`, `Integer` and `Double`:

- `Int` corresponds to the plain integer type found in most languages. It has fixed maximum and minimum values that depend on a computer's processor. (In 32-bit machines the range goes from -2147483648 to 2147483647).
- `Integer` also is used for integer numbers, but it supports arbitrarily large values – at the cost of some efficiency.
- `Double` is the double-precision floating point type, a good choice for real numbers in the vast majority of cases. (Haskell also has `Float`, the single-precision counterpart of `Double`, which is usually less attractive due to further loss of precision.)

Several other number types are available, but these cover most in everyday tasks.

7.2.1 Polymorphic guesswork

If you've read carefully this far, you know that we don't need to specify types always because the compiler can *infer* types. You also know that we cannot mix types when functions require matched types. Combine this with our new understanding of numbers to understand how Haskell handles basic arithmetic like this:

[4] This is a loose definition, but will suffice until we discuss typeclasses in more detail.

```
Prelude> (-7) + 5.12
-1.88
```

This may seem to add two numbers of different types – an integer and a non-integer. Let's see what the types of the numbers we entered actually are:

```
Prelude> :t (-7)
(-7) :: (Num a) => a
```

So, `(-7)` is neither `Int` nor `Integer`! Rather, it is a *polymorphic constant*, which can ”morph” into any number type. Now, let's look at the other number:

```
Prelude> :t 5.12
5.12 :: (Fractional t) => t
```

`5.12` is also a polymorphic constant, but one of the `Fractional` class, which is a subset of `Num` (every `Fractional` is a `Num`, but not every `Num` is a `Fractional`; for instance, `Int`s and `Integer`s are not `Fractional`).

When a Haskell program evaluates `(-7) + 5.12`, it must settle for an actual matching type for the numbers. The type inference accounts for the class specifications: `(-7)` can be any `Num`, but there are extra restrictions for `5.12`, so that's the limiting factor. With no other restrictions, `5.12` will assume the default `Fractional` type of `Double`, so `(-7)` will become a `Double` as well. Addition then proceeds normally and returns a `Double`.[5][6]

The following test will give you a better feel of this process. In a source file, define

```
x = 2
```

Then load the file in GHCi and check the type of `x`. Then, change the file to add a `y` variable,

```
x = 2
y = x + 3
```

reload it and check the types of `x` and `y`. Finally, modify `y` to

```
x = 2
y = x + 3.1
```

and see what happens with the types of both variables.

[5] For seasoned programmers:

[6] This appears to have the same effect that programs in C (and many other languages) manage with an *implicit cast* (where an integer literal is silently converted to a double). In C, however, the conversion is done behind your back, while in Haskell it only occurs if the variable/literal is a polymorphic constant. This distinction will become clearer shortly, when we show a counter-example.

7.2.2 Monomorphic trouble

The sophistication of the numerical types and classes occasionally leads to some complications. Consider, for instance, the common division operator `(/)`. It has the following type signature:

```
(/) :: (Fractional a) => a -> a -> a
```

Restricting `a` to fractional types is a must because the division of two integer numbers will often result in a non-integer. Nevertheless, we can still write something like

```
Prelude> 4 / 3
1.3333333333333333
```

because the literals `4` and `3` are polymorphic constants and therefore assume the type `Double` at the behest of `(/)`. Suppose, however, we want to divide a number by the length of a list.[7] The obvious thing to do would be using the `length` function:

```
Prelude> 4 / length [1,2,3]
```

Unfortunately, that blows up:

```
<interactive>:1:0:
      No instance for (Fractional Int)
        arising from a use of `/' at <interactive>:1:0-17
      Possible fix: add an instance declaration for (Fractional Int)
      In the expression: 4 / length [1, 2, 3]
      In the definition of `it': it = 4 / length [1, 2, 3]
```

As usual, the problem can be understood by looking at the type signature of `length`:

```
length :: [a] -> Int
```

The result of `length` is an `Int`, not a polymorphic constant. As an `Int` is not a `Fractional`, Haskell won't let us use it with `(/)`.

To escape this problem, we have a special function. Before following on with the text, try to guess what this does only from the name and signature:

```
fromIntegral :: (Integral a, Num b) => a -> b
```

`fromIntegral` takes an argument of some `Integral` type (like `Int` or `Integer`) and makes it a polymorphic constant. By combining it with `length`, we can make the length of the list fit into the signature of `(/)`:

[7] A reasonable scenario – think of computing an average of the values in a list.

```
Prelude> 4 / fromIntegral (length [1,2,3])
1.3333333333333333
```

In some ways, this issue is annoying and tedious, but it is an inevitable side-effect of having a rigorous approach to manipulating numbers. In Haskell, if you define a function with an `Int` argument, it will never be converted to an `Integer` or `Double`, unless you explicitly use a function like `fromIntegral`. As a direct consequence of its refined type system, Haskell has a surprising diversity of classes and functions dealing with numbers.

7.3 Classes beyond numbers

Haskell has typeclasses beyond arithmetic. For example, the type signature of `(==)` is:

```
(==) :: (Eq a) => a -> a -> Bool
```

Like `(+)` or `(/)`, `(==)` is a polymorphic function. It compares two values of the same type, which must belong to the class `Eq` and returns a `Bool`. `Eq` is simply the class for types of values which can be compared for equality, and it includes all of the basic non-functional types.[8]

Typeclasses add a lot to the power of the type system. We will return to this topic later to see how to use them in custom ways.

[8] Comparing two functions for equality is considered intractable

8 Building vocabulary

This chapter will be a bit of an interlude with some advice for studying and using Haskell. We will discuss the importance of acquiring a vocabulary of functions and how this book and other resources can help. First, however, we need to understand function composition.

8.1 Function composition

Function composition means applying one function to a value and then applying another function to the result. Consider these two functions:

Example: Simple functions

```
f x = x + 3

square x = x  2
```

We can compose them in two different ways, depending on which one we apply first:

```
Prelude> square (f 1)
16
Prelude> square (f 2)
25
Prelude> f (square 1)
4
Prelude> f (square 2)
7
```

The parentheses around the inner function are necessary; otherwise, the interpreter would think that you were trying to get the value of `square f`, or `f square`; and both of those would give type errors.

The composition of two functions results in a function in its own right. If we regularly apply f and then square (or vice-versa), we should generate a new variable name for the resulting combinations:

Example: Composed functions

```
squareOfF x = square (f x)

fOfSquare x = f (square x)
```

There is a second, nifty way of writing composed functions. It uses `(.)`, the function composition operator and is as simple as putting a period between the two functions:

Example: Composing functions with `(.)`

```
squareOfF x = (square . f) x

fOfSquare x = (f . square) x
```

Note that functions are still applied from right to left, so that `g(f(x)) == (g . f) x`. `(.)` is modeled after the mathematical operator ∘, which works in the same way: $(g \circ f)(x) = g(f(x))$.

Incidentally, our function definitions *are* effectively mathematical equations, so we can take

```
squareOfF x = (square . f) x
```

and cancel the `x` from both sides, leaving:

```
squareOfF = square . f
```

We will later learn more about such cases of functions without arguments shown. For now, understand we can simply substitute our defined variable name for any case of the composed functions.

8.2 The need for a vocabulary

Haskell makes it simple to write composed functions and to define variables, so we end up with relatively simple, elegant, and expressive code. Of course, to use function composition, we first need to have functions to compose. While functions we write ourselves will always be available, every installation of GHC comes with a vast assortment of libraries (i.e. packaged code), which provide functions for many common tasks. For that reason, effective Haskell programmers need some familiarity with the essential libraries. At the least, you should know how to find useful functions in the libraries when you need them.

Given only the Haskell syntax we will cover through the Recursion[1] chapter, we will, in principle, have enough knowledge to write nearly any list manipulation program we want.

[1] Chapter 12 on page 69

However, writing full programs with only these basics would be terribly inefficient because we would end up rewriting large parts of the standard libraries. So, much of our study going forward will involve studying and understanding these valuable tools the Haskell community has already built.

8.3 Prelude and the libraries

Here are a few basic facts about Haskell libraries:

First and foremost, *Prelude* is the core library loaded by default in every Haskell program. Alongside with the basic types, it provides a set of ubiquitous and useful functions. We will refer to Prelude and its functions all the time throughout these introductory chapters.

GHC includes a large set of core libraries that provide a wide range of tools, but only Prelude is loaded automatically. The other libraries are available as *modules* which you can *import* into your program. Later on, we will explain the minutiae of how modules work. For now, just know that your source file needs lines near the top to import any desired modules. For example, to use the function `permutations` from the module `Data.List`, add the line `import Data.List` to the top of your .hs file. Here's a full source file example:

Example: Importing a module in a source file

```
import Data.List


testPermutations = permutations "Prelude"
```

For quick GHCi tests, just enter `:m +Data.List` at the command line to load that module.

```
Prelude> :m +Data.List
Prelude Data.List> :t permutations
permutations :: [a] -> a[2]
```

8.4 One exhibit

Before continuing, let us see one (slightly histrionic, we admit) example of what familiarity with a few basic functions from Prelude can bring us.[3] Suppose we need a function which takes a string composed of words separated by spaces and returns that string with the order of the words reversed, so that `"Mary had a little lamb"` becomes `"lamb little a had Mary"`. We *could* solve this problem using only the basics we have already covered along with a few insights in the upcoming Recursion chapter. Below is one messy, complicated solution. Don't stare at it for too long!

[3] The example here is inspired by the Simple Unix tools ^{`http://www.haskell.org/haskellwiki/Simple_unix_tools`} demo in the HaskellWiki.

Example: There be dragons

```
monsterRevWords :: String -> String
monsterRevWords input = rejoinUnreversed (divideReversed input)
    where
    divideReversed s = go1 [] s
        where
        go1 divided [] = divided
        go1 [] (c:cs)
            | testSpace c = go1 [] cs
            | otherwise   = go1 [[]] (c:cs)
        go1 (w:ws) [c]
            | testSpace c = (w:ws)
            | otherwise   = ((c:w):ws)
        go1 (w:ws) (c:c':cs)
            | testSpace c =
                if testSpace c'
                    then go1 (w:ws) (c':cs)
                    else go1 ([c']:w:ws) cs
            | otherwise = go1 ((c:w):ws) (c':cs)
    testSpace c = c == ' '
    rejoinUnreversed [] = []
    rejoinUnreversed [w] = reverseList w
    rejoinUnreversed strings = go2 (' ' : reverseList newFirstWord) (otherWords)
        where
        (newFirstWord : otherWords) = reverseList strings
        go2 rejoined ([]:[]) = rejoined
        go2 rejoined ([]:(w':ws')) = go2 (rejoined) ((' ':w'):ws')
        go2 rejoined ((c:cs):ws) = go2 (c:rejoined) (cs:ws)
    reverseList [] = []
    reverseList w = go3 [] w
        where
        go3 rev [] = rev
        go3 rev (c:cs) = go3 (c:rev) cs
```

There are too many problems with this *thing*; so let us consider just three of them:

- To see whether `monsterRevWords` does what you expect, you could either take our word for it, test it exhaustively on all sorts of possible inputs, or attempt to understand it and get an awful headache (please don't).
- Furthermore, if we write a function this ugly and have to fix a bug or slightly modify it later on,[4] we are set for an awful time.

[4] *Co-author's note*: ”Later on? I wrote that half an hour ago, and I'm not totally sure about how it works already...”

- Finally, we have at least one easy-to-spot potential problem: if you have another glance at the definition, about halfway down there is a `testSpace` helper function which checks if a character is a space or not. The test, however, only includes the common space character (that is, `' '`), and no other whitespace characters (tabs, newlines, etc.).[5]

We can do much better than the junk above if we use the following Prelude functions:

- `words`, which reliably breaks down a string in whitespace delimited words, returning a list of strings;
- `reverse`, which reverses a list (incidentally, that is exactly what the `reverseList` above does); and
- `unwords`, which does the opposite of `words`;

then function composition means our problem is instantly solved.

Example: `revWords` done the Haskell way

```
revWords :: String -> String
revWords input = (unwords . reverse . words) input
```

That's short, simple, readable and (since Prelude is reliable) bug-free.[6] So, any time some program you are writing begins to look like `monsterRevWords`, look around and reach for your toolbox — the libraries.

8.5 This book's use of the libraries

After the stern warnings above, you might expect us to continue diving deep into the standard libraries. However, the Beginner's Track is meant to cover Haskell functionality in a conceptual, readable, and reasonably compact manner. A systematic study of the libraries would not help us, but we will introduce functions from the libraries as appropriate to each concept we cover.

- In the Elementary Haskell section, several of the exercises (mainly, among those about list processing) involve writing equivalent definitions for Prelude functions. For each of these exercises you do, one more function will be added to your repertoire.
- Every now and then we will introduce more library functions; maybe within an example, or just with a mention in passing. Whenever we do so, take a minute to test the function and do some experiments. Remember to extend that habitual curiosity about types we mentioned in Type basics[7] to the functions themselves.

[5] A reliable way of checking whether a character is whitespace is with the `isSpace` function, which is in the module `Data.Char`.

[6] In case you are wondering, many other functions from Prelude or `Data.List` could help to make `monsterRevWords` somewhat saner — to name a few: `(++)`, `concat`, `groupBy`, `intersperse` — but no use of those would compare to the one-liner above.

[7] Chapter 5 on page 25

- While the first few chapters are quite tightly-knit, later parts of the book are more independent. Haskell in Practice includes chapters on the Hierarchical libraries[8], and most of their content can be understood soon after having completed Elementary Haskell.
- As we reach the later parts of the Beginner's track, the concepts we will discuss (monads in particular) will naturally lead to exploration of important parts of the core libraries.

8.6 Other resources

- First and foremost, all modules have basic documentation. You may not be ready to read that directly yet, but we'll get there. You can read the Prelude specification[9] on-line as well as the documentation of the libraries bundled with GHC[10], with nice navigation and source code just one click away.
- Hoogle[11] is a great way to search through the documentation. It is a Haskell search engine which covers the core libraries. You can search for everything from function names to type definitions and more.
- Beyond the libraries included with GHC, there is a large ecosystem of libraries, made available through Hackage[12] and installable with a tool called cabal[13]. The Hackage site has documentation for its libraries. We will not venture outside of the core libraries in the Beginner's Track, but you should certainly use Hackage once you begin your own projects. A second Haskell search engine called Hayoo![14] covers all of Hackage.
- When appropriate, we will give pointers to other useful learning resources, especially when we move towards intermediate and advanced topics.

[8] Chapter 70 on page 493

[9] `http://www.haskell.org/onlinereport/standard-prelude.html`

[10] `http://www.haskell.org/ghc/docs/latest/html/libraries/index.html`

[11] `http://www.haskell.org/hoogle`

[12] `https://hackage.haskell.org/`

[13] `http://www.haskell.org/cabal/users-guide/`

[14] `http://holumbus.fh-wedel.de/hayoo/hayoo.html`

9 Next steps

This chapter introduces *pattern matching* and two new pieces of syntax: `if` expressions and `let` bindings.

9.1 if / then / else

Haskell syntax supports garden-variety conditional expressions of the form *if... then... else ...*. For instance, consider a function that returns `(-1)` if its argument is less than `0`; `0` if its argument *is* `0`; and `1` if its argument is greater than `0`. The predefined `signum` function does that job already; but for the sake of illustration, let's define a version of our own:

Example: The signum function.

```
mySignum x =
    if x < 0
        then -1
        else if x > 0
            then 1
            else 0
```

You can experiment with this:

- `Main> mySignum 5 1`
- `Main> mySignum 0 0`
- `Main> mySignum (5 - 10) -1`
- `Main> mySignum (-1) -1`

The parentheses around "-1" in the last example are required; if missing, Haskell will think that you are trying to subtract `1` from `mySignum` (which would give a type error).

In an if/then/else construct, first the condition (in this case `x < 0`) is evaluated. If it results `True`, the whole construct evaluates to the **`then`** expression; otherwise (if the condition is `False`), the construct evaluates to the **`else`** expression. All of that is pretty intuitive. If you have programmed in an imperative language before, however, it might seem surprising to know that Haskell always requires *both* a **`then`** *and* an **`else`** clause. The construct has to result in a value in both cases and, specifically, a value of the same type in both cases.

Function definitions using if / then / else like the one above can be rewritten using Guards[1].

[1] Chapter 4.5 on page 21

Example: From if to guards

```
mySignum x
    | x < 0     = -1
    | x > 0     = 1
    | otherwise = 0
```

Similarly, the absolute value function defined in Truth values[2] can be rendered with an if/then/else:

Example: From guards to if

```
absolute x =
    if x < 0
        then -x
        else x
```

Why use if/then/else versus guards? As you will see with later examples and in your own programming, either way of handling conditionals may be more readable or convenient depending on the circumstances. In many cases, both options work equally well.

9.2 Introducing pattern matching

Consider a program which tracks statistics from a racing competition in which racers receive points based on their classification in each race, the scoring rules being:

- 10 points for the winner;
- 6 for second-placed;
- 4 for third-placed;
- 3 for fourth-placed;
- 2 for fifth-placed;
- 1 for sixth-placed;
- no points for other racers.

We can write a simple function which takes a classification (represented by an integer number: `1` for first place, etc.[3]) and returns how many points were earned. One possible solution uses if/then/else:

[2] Chapter 4.5 on page 21

[3] Here we will not be much worried about what happens if a nonsensical value (say, `(-4)`) is passed to the function. In general, however, it is a good idea to give some thought to such ”strange” cases, in order to avoid nasty surprises down the road.

Example: Computing points with if/then/else

```
pts :: Int -> Int
pts x =
    if x == 1
        then 10
        else if x == 2
            then 6
            else if x == 3
                then 4
                else if x == 4
                    then 3
                    else if x == 5
                        then 2
                        else if x == 6
                            then 1
                            else 0
```

Yuck! Admittedly, it wouldn't look this hideous had we used guards instead of if/then/else, but it still would be tedious to write (and read!) all those equality tests. We can do better, though:

Example: Computing points with a piece-wise definition

```
pts :: Int -> Int
pts 1 = 10
pts 2 = 6
pts 3 = 4
pts 4 = 3
pts 5 = 2
pts 6 = 1
pts _ = 0
```

Much better. However, even though defining `pts` in this style (which we will arbitrarily call *piece-wise definition* from now on) shows to a reader of the code what the function does in a clear way, the syntax looks odd given what we have seen of Haskell so far. Why are there seven equations for `pts`? What are those numbers doing in their left-hand sides? What about variable arguments?

This feature of Haskell is called *pattern matching*. When we call `pts`, the argument is *matched* against the numbers on the left side of each of the equations, which in turn are the *patterns*. The matching is done in the order we wrote the equations. First, the argument is matched against the `1` in the first equation. If the argument is indeed `1`, we have a match and the first equation is used; so `pts 1` evaluates to `10` as expected. Otherwise, the other equations are tried in order following the same procedure. The final one, though, is rather different: the _ is a special pattern, often called a ”wildcard”, that might be read as ”whatever”: it matches with anything; and therefore if the argument doesn’t match any of the previous patterns pts will return zero.

As for the lack of x or any other variable standing for the argument, we simply don’t need that to write the definitions. All possible return values are constants. Besides, variables are used to express relationships on the right side of the definition, so the x is unnecessary in our pts function.

However, we could use a variable to make pts even more concise. The points given to a racer decrease regularly from third place to sixth place, at a rate of one point per position. After noticing that, we can eliminate three of the seven equations as follows:

Example: Mixing styles

```
pts :: Int -> Int
pts 1 = 10
pts 2 = 6
pts x
    | x <= 6    = 7 - x
    | otherwise = 0
```

So, we can mix both styles of definitions. In fact, when we write pts x in the left side of an equation we are using pattern matching too! As a pattern, the x (or any other variable name) matches anything just like _; the only difference being that it also gives us a name to use on the right side (which, in this case, is necessary to write 7 - x).

Exercises:
We cheated a little when moving from the second version of pts to the third one: they do not do exactly the same thing. Can you spot what the difference is?

Beyond integers, pattern matching works with values of various other types. One handy example is booleans. For instance, the (||) logical-or operator we met in Truth values[4] could be defined as:

Example: (||)

```
(||) :: Bool -> Bool -> Bool
False || False = False
_     || _     = True
```

Or:

[4] Chapter 4.5.1 on page 22

Example: `(||)`, done another way

```
(||) :: Bool -> Bool -> Bool
True  || _ = True
False || y = y
```

When matching two or more arguments at once, the equation will only be used if all of them match.

Now, let's discuss a few things that might go wrong when using pattern matching:

- If we put a pattern which matches anything (such as the final patterns in each of the `pts` example) *before* the more specific ones the latter will be ignored. GHC(i) will typically warn us that "Pattern match(es) are overlapped" in such cases.
- If no patterns match, an error will be triggered. Generally, it is a good idea to ensure the patterns cover all cases, in the same way that the `otherwise` guard is not mandatory but highly recommended.
- Finally, while you can play around with various ways of (re)defining `(&&)`,[5] here is one version that will *not* work:

```
(&&) :: Bool -> Bool -> Bool
x && x = x -- oops!
_ && _ = False
```

The program won't test whether the arguments are equal just because we happened to use the same name for both. As far as the matching goes, we could just as well have written `_ && _` in the first case. And even worse: because we gave the same name to both arguments, GHC(i) will refuse the function due to "Conflicting definitions for 'x'".

9.3 Tuple and list patterns

While the examples above show that pattern matching helps in writing more elegant code, that does not explain why it is so important. So, let's consider the problem of writing a definition for `fst`, the function which extracts the first element of a pair. At this point, that appears to be an impossible task, as the only way of accessing the first value of the pair is by using `fst` itself... The following function, however, does the same thing as `fst` (confirm it in GHCi):

Example: A definition for `fst`

```
fst' :: (a, b) -> a
fst' (x, _) = x
```

[5] If you are going to experiment with it in GHCi, call your version something else to avoid a name clash; say, (&!&).

It's magic! Instead of using a regular variable in the left side of the equation, we specified the argument with the *pattern* of the 2-tuple - that is, `(,)` - filled with a variable and the `_` pattern. Then the variable was automatically associated with the first component of the tuple, and we used it to write the right side of the equation. The definition of `snd` is, of course, analogous.

Furthermore, the trick demonstrated above can be done with lists as well. Here are the actual definitions of `head` and `tail`:

Example: `head`, `tail` and patterns

```
head             :: [a] -> a
head (x:_)       =  x
head []          =  error "Prelude.head: empty list"

tail             :: [a] -> [a]
tail (_:xs)      =  xs
tail []          =  error "Prelude.tail: empty list"
```

The only essential change in relation to the previous example was replacing `(,)` with the pattern of the cons operator `(:)`. These functions also have an equation using the pattern of the empty list, `[]`; however, since empty lists have no head or tail there is nothing to do other than use `error` to print a prettier error message.

In summary, the power of pattern matching comes from its use in accessing the parts of a complex value. Pattern matching on lists, in particular, will be extensively deployed in Recursion[6] and the chapters that follow it. Later on, we will explore what is happening behind this seemingly magical feature.

9.4 `let` bindings

To conclude this chapter, a brief word about `let` bindings (an alternative to `where` clauses for making local declarations). For instance, take the problem of finding the roots of a polynomial of the form ax^2+bx+c (in other words, the solution to a second degree equation — think back to your middle school math courses). Its solutions are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We could write the following function to compute the two values of x:

```
roots a b c =
    ((-b + sqrt(b * b - 4 * a * c)) / (2 * a),
     (-b - sqrt(b * b - 4 * a * c)) / (2 * a))
```

[6] Chapter 12 on page 69

Writing the sqrt(b * b - 4 * a * c) term in both cases is annoying, though; we can use a local binding instead, using either where or, as will be demonstrated below, a let declaration:

```
roots a b c =
    let sdisc = sqrt (b * b - 4 * a * c)
    in  ((-b + sdisc) / (2 * a),
         (-b - sdisc) / (2 * a))
```

We put the let keyword before the declaration, and then use in to signal we are returning to the ”main” body of the function. It is possible to put multiple declarations inside a single let...in block — just make sure they are indented the same amount or there will be syntax errors:

```
roots a b c =
    let sdisc = sqrt (b * b - 4 * a * c)
        twice_a = 2 * a
    in  ((-b + sdisc) / twice_a,
         (-b - sdisc) / twice_a)
```

⚠ Warning

Because indentation matters syntactically in Haskell, you need to be careful about whether you are using tabs or spaces. By far the best solution is to configure your text editor to insert two or four spaces in place of tabs. If you insist on keeping tabs as distinct, at least ensure that your tabs always have the same length.

Note:
The Indentation[a] chapter has a full account of indentation rules.

[a] Chapter 23 on page 133

10 Simple input and output

10.1 Back to the real world

Beyond internally calculating values, we want our programs to interact with the world. The most common beginners' program in any language simply displays a ”hello world” greeting on the screen. Here's a Haskell version:

```
Prelude> putStrLn "Hello, World!"
```

putStrLn is one of the standard Prelude tools. As the ”putStr” part of the name suggests, it takes a String as an argument and prints it to the screen. We could use putStr on its own, but we usually include the ”Ln” part so to also print a line break. Thus, whatever else is printed next will appear on a new line.

So now you should be thinking, ”what is the type of the putStrLn function?” It takes a String and gives... um... what? What do we call that? The program doesn't get something back that it can use in another function. Instead, the result involves having the computer change the screen. In other words, it does something in the world outside of the program. What type could that have? Let's see what GHCi tells us:

```
Prelude> :t putStrLn
putStrLn :: String -> IO ()
```

”IO” stands for ”input and output”. Wherever there is IO in a type, interaction with the world outside the program is involved. We'll call these IO values *actions*. The other part of the IO type, in this case (), is the type of the return value of the action; that is, the type of what it gives back to the program (as opposed to what it does outside the program). () (pronounced as ”unit”) is a type that only contains one value also called () (effectively a tuple with zero elements). Since putStrLn sends output to the world but doesn't return anything to the program, () is used as a placeholder. We might read IO () as ”action which returns ()”.

A few more examples of when we use IO:

- print a string to the screen
- read a string from a keyboard
- write data to a file
- read data from a file

What makes IO actually work? Lots of things happen behind the scenes to take us from putStrLn to pixels in the screen, but we don't need to understand any of the details to write our programs. A complete Haskell program is actually a big IO action. In a compiled program, this action is called main and has type IO (). From this point of view, to write a Haskell program is to combine actions and functions to form the overall action main that will be executed when the program is run. The compiler takes care of instructing the computer on how to do this.

Exercises:
Back in the Type Basics chapter, we mentioned that the type of the `openWindow` function[a] had been simplified. Can you guess what the simplification was?

[a] Chapter 5.1.1 on page 26

10.2 Sequencing actions with *do*

do notation provides a convenient means of putting actions together (which is essential in doing useful things with Haskell). Consider the following program:

Example: What is your name?

```
main = do
  putStrLn "Please enter your name:"
  name <- getLine
  putStrLn ("Hello, " ++ name ++ ", how are you?")
```

Note:
Even though `do` notation looks very different from the Haskell code we have seen so far, it is just syntactic sugar for a handful of functions, the most important of them being the `(>>=)` operator. We could explain how those functions work and *then* introduce `do` notation. However, there are several topics we would need to cover before we can give a convincing explanation. Jumping in with `do` right now is a pragmatic short cut that will allow you to start writing complete programs with IO right away. We will see how `do` works later in the book, beginning with the ../Understanding monads/[a] chapter.

[a] Chapter 30 on page 179

Before we get into how *do* works, take a look at `getLine`. It goes to the outside world (to the terminal in this case) and brings back a `String`. What is its type?

```
Prelude> :t getLine
getLine :: IO String
```

That means `getLine` is an IO action that, when run, will return a `String`. But what about the input? While functions have types like `a -> b` which reflect that they take arguments and give back results, `getLine` doesn't actually take an argument. It takes as input whatever is in the line in the terminal. However, that line in the outside world isn't a defined value with a type until we bring it into the Haskell program.

The program doesn't know the state of the outside world until runtime, so it cannot predict the exact results of IO actions. To manage the relationship of these IO actions to other aspects of a program, the actions must be executed in a predictable sequence defined in advance in the code. With regular functions that do not perform IO, the exact sequencing of execution is less of an issue — as long as the results eventually go to the right places.

In our name program, we're sequencing three actions: a `putStrLn` with a greeting, a `getLine`, and another `putStrLn`. With the `getLine`, we use `<-` notation which assigns a variable name to stand for the returned value. We cannot know what the value will be in advance, but we know it will use the specified variable name, so we can then use the variable elsewhere (in this case, to prepare the final message being printed). The final action defines the type of the whole `do` block. Here, the final action is the result of a `putStrLn`, and so our whole program has type `IO ()`.

Exercises:
Write a program which asks the user for the base and height of a right angled triangle, calculates its area, and prints it to the screen. The interaction should look something like:

```
The base?
3.3
The height?
5.4
The area of that triangle is 8.91
```

You will need to use the function `read` to convert user strings like "3.3" into numbers like 3.3 and the function `show` to convert a number into string.

10.2.1 Left arrow clarifications

While actions like `getLine` are almost always used to get values, we are not obliged to actually get them. For example, we could write something like this:

Example: executing `getLine` directly

```
main = do
  putStrLn "Please enter your name:"
  getLine
  putStrLn "Hello, how are you?"
```

In this case, we don't use the input at all, but we still give the user the experience of entering their name. By omitting the `<-`, the action will happen, but the data won't be stored or accessible to the program.

`<-` can be used with any action except the last

There are very few restrictions on which actions can have values obtained from them. Consider the following example where we put the results of each action into a variable (except the last... more on that later):

Example: putting all results into a variable

```
main = do
  x <- putStrLn "Please enter your name:"
  name <- getLine
  putStrLn ("Hello, " ++ name ++ ", how are you?")
```

The variable `x` gets the value out of its action, but that isn't useful in this case because the action returns the unit value `()`. So while we could technically get the value out of any action, it isn't always worth it.

So, what about the final action? Why can't we get a value out of that? Let's see what happens when we try:

Example: getting the value out of the last action

```
main = do
  x <- putStrLn "Please enter your name:"
  name <- getLine
  y <- putStrLn ("Hello, " ++ name ++ ", how are you?")
```

Whoops! Error!

```
  HaskellWikibook.hs:5:2:
        The last statement in a 'do' construct must be an expression
```

Making sense of this requires a somewhat deeper understanding of Haskell than we currently have. Suffice it to say, whenever you use `<-` to get the value of an action, Haskell is always expecting another action to follow it. So the final action cannot have any `<-`s.

10.2.2 Controlling actions

Normal Haskell constructions like **`if/then/else`**can be used within the **`do`** notation, but you need to take some care here. For instance, in a simple ”guess the number” program, we have:

```
doGuessing num = do
   putStrLn "Enter your guess:"
   guess <- getLine
   if (read guess) < num
     then do putStrLn "Too low!"
             doGuessing num
     else if (read guess) > num
            then do putStrLn "Too high!"
```

```
                doGuessing num
        else putStrLn "You Win!"
```

Remember that the **if/then/else** construction takes three arguments: the condition, the ”then” branch, and the ”else” branch. The condition needs to have type `Bool`, and the two branches can have any type, provided that they have the *same* type. The type of the entire **if/then/else**construction is then the type of the two branches.

In the outermost comparison, we have `(read guess) < num` as the condition. That has the correct type. Let’s now consider the ”then” branch. The code here is:

```
do putStrLn "Too low!"
   doGuessing num
```

Here, we are sequencing two actions: `putStrLn` and `doGuessing`. The first has type `IO ()`, which is fine. The second also has type `IO ()`, which is fine. The type result of the entire computation is precisely the type of the final computation. Thus, the type of the ”then” branch is also `IO ()`. A similar argument shows that the type of the ”else” branch is also `IO ()`. This means the type of the entire **if/then/else**construction is `IO ()`, which is what we want.

Note: be careful if you find yourself thinking, ”Well, I already started a **do** block; I don’t need another one.” We can’t have code like:

```
do if (read guess) < num
     then putStrLn "Too low!"
          doGuessing num
     else ...
```

Here, since we didn’t repeat the **do**, the compiler doesn’t know that the `putStrLn` and `doGuessing` calls are supposed to be sequenced, and the compiler will think you’re trying to call `putStrLn` with three arguments: the string, the function `doGuessing` and the integer `num`, and thus reject the program.

Exercises:
Write a program that asks the user for his or her name. If the name is one of Simon, John or Phil, tell the user that you think Haskell is a great programming language. If the name is Koen, tell them that you think debugging Haskell is fun (Koen Classen is one of the people who works on Haskell debugging); otherwise, tell the user that you don’t know who he or she is. (As far as syntax goes there are a few different ways to do it; write at least a version using `if` / `then` / `else`.)

10.3 Actions under the microscope

Actions may look easy up to now, but they are a common stumbling block for new Haskellers. If you have run into trouble working with actions, see if any of your problems or questions match any of the cases below. We suggest skimming this section now, then come back here when you actually experience trouble.

10.3.1 Mind your action types

One temptation might be to simplify our program for getting a name and printing it back out. Here is one unsuccessful attempt:

Example: Why doesn't this work?

```
main =
 do putStrLn "What is your name? "
    putStrLn ("Hello " ++ getLine)
```

Ouch! Error!

```
  HaskellWikiBook.hs:3:26:

        Couldn't match expected type `[Char]'
               against inferred type `IO String'
```

Let us boil the example above down to its simplest form. Would you expect this program to compile?

Example: This still does not work

```
main =
 do putStrLn getLine
```

For the most part, this is the same (attempted) program, except that we've stripped off the superfluous ”What is your name” prompt as well as the polite ”Hello”. One trick to understanding this is to reason about it in terms of types. Let us compare:

```
putStrLn :: String -> IO ()
getLine  :: IO String
```

We can use the same mental machinery we learned in ../Type basics/[1] to figure how this went wrong. `putStrLn` is expecting a `String` as input. We do not have a `String`; we have something tantalisingly close: an `IO String`. This represents an action that will *give* us a `String` when it's run. To obtain the `String` that `putStrLn` wants, we need to run the action, and we do that with the ever-handy left arrow, `<-`.

[1] Chapter 5 on page 25

Example: This time it works

```
main =
 do name <- getLine
    putStrLn name
```

Working our way back up to the fancy example:

```
main =
 do putStrLn "What is your name? "
    name <- getLine
    putStrLn ("Hello " ++ name)
```

Now the name is the String we are looking for and everything is rolling again.

10.3.2 Mind your expression types too

So, we've made a big deal out of the idea that you can't use actions in situations that don't call for them. The converse of this is that you can't use non-actions in situations that expect actions. Say we want to greet the user, but this time we're so excited to meet them, we just have to SHOUT their name out:

Example: Exciting but incorrect. Why?

```
import Data.Char (toUpper)

main =
 do name <- getLine
    loudName <- makeLoud name
    putStrLn ("Hello " ++ loudName ++ "!")
    putStrLn ("Oh boy! Am I excited to meet you, " ++ loudName)

-- Don't worry too much about this function; it just converts a String to
 uppercase
makeLoud :: String -> String
makeLoud s = map toUpper s
```

This goes wrong...

```
        Couldn't match expected type `IO' against inferred type `[]'
          Expected type: IO t
          Inferred type: String
        In a 'do' expression: loudName <- makeLoud name
```

This is similar to the problem we ran into above: we've got a mismatch between something expecting an IO type and something which does not produce IO. This time, the trouble is the left arrow `<-`; we're trying to left-arrow a value of `makeLoud name`, which really isn't left arrow material. It's basically the same mismatch we saw in the previous section, except now we're trying to use regular old String (the loud name) as an IO String. The latter is an action, something to be run, whereas the former is just an expression minding its own business. We cannot simply use `loudName = makeLoud name` because a `do` sequences *actions*, and `loudName = makeLoud name` is not an action.

So how do we extricate ourselves from this mess? We have a number of options:

- We could find a way to turn `makeLoud` into an action, to make it return `IO String`. However, we don't want to make actions go out into the world for no reason. Within our program, we can reliably verify how everything is working. When actions engage the outside world, our results are much less predictable. An IO `makeLoud` would be misguided. Consider another issue too: what if we wanted to use makeLoud from some other, non-IO, function? We really don't want to engage IO actions except when absolutely necessary.
- We could use a special code called `return` to promote the loud name into an action, writing something like `loudName <- return (makeLoud name)`. This is slightly better. We at least leave the `makeLoud` function itself nice and IO-free whilst using it in an IO-compatible fashion. That's still moderately clunky because, by virtue of left arrow, we're implying that there's action to be had -- how exciting! -- only to let our reader down with a somewhat anticlimactic `return` (note: we will learn more about appropriate uses for `return` in later chapters).

- Or we could use a let binding...

It turns out that Haskell has a special extra-convenient syntax for let bindings in actions. It looks a little like this:

Example: `let` bindings in `do` blocks.

```
main =
 do name <- getLine
    let loudName = makeLoud name
    putStrLn ("Hello " ++ loudName ++ "!")
    putStrLn ("Oh boy! Am I excited to meet you, " ++ loudName)
```

If you're paying attention, you might notice that the let binding above is missing an `in`. This is because `let` bindings inside `do` blocks do not require the `in` keyword. You could very well use it, but then you'd have messy extra do blocks. For what it's worth, the following two blocks of code are equivalent.

sweet	unsweet

`do name <- getLine`
`   let loudName = makeLoud name`
`   putStrLn ("Hello " ++ loudName ++ "!")`
`   putStrLn (`
`       "Oh boy! Am I excited to meet you, "`
`           ++ loudName)`
	`do name <- getLine`
	`   let loudName = makeLoud name`
	`   in  do putStrLn ("Hello " ++ loudName ++ "!")`
	`          putStrLn (`
	`              "Oh boy! Am I excited to meet you, "`
	`                  ++ loudName)`

Exercises:

1. Why does the unsweet version of the let binding require an extra `do` keyword?
2. Do you always need the extra `do`?
3. (extra credit) Curiously, `let` without `in` is exactly how we wrote things when we were playing with the interpreter in the beginning of this book. Why is it ok to omit the `in` keyword in the interpreter but needed (outside of *do* blocks) in a source file?

10.4 Learn more

At this point, you have the fundamentals needed to do some fancier input/output. Here are some IO-related topics you may want to check in parallel with the main track of this course.

- You could continue the sequential track, learning more about types[2] and eventually monads[3].
- Alternately: you could start learning about building graphical user interfaces in the ../GUI/[4] chapter
- For more IO-related functionality, you could also consider learning more about the System.IO library[5]

[2] Chapter 15 on page 95
[3] Chapter 30 on page 179
[4] Chapter 85 on page 557
[5] Chapter 75 on page 505

11 Elementary Haskell

12 Recursion

Recursion plays a central role in Haskell (and computer science and mathematics in general). Recursion is merely a form of repetition, but sometimes it is taught in confusing or obscure ways. To understand recursion, you should separate the *meaning* of a recursive function from its *behaviour*.

A function is recursive when one part of its definition includes the function itself again. Along with the recursive condition, these functions generally also contain at least one *base case* condition that stops (i.e. *terminates*) the function without calling the function again. Without a terminating condition, recursive functions would lead to infinite regress (i.e. an infinite loop).

12.1 Numeric recursion

12.1.1 The factorial function

Mathematics (specifically combinatorics) has a function called **factorial**.[1] It takes a single non-negative integer as an argument, finds all the positive integers less than or equal to ”n”, and multiplies them all together. For example, the factorial of 6 (denoted as 6!) is $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. We can use a recursive style to define this in Haskell:

Let's look at the factorials of two adjacent numbers:

Example: Factorials of consecutive numbers

```
Factorial of 6 = 6 × 5 × 4 × 3 × 2 × 1
Factorial of 5 =     5 × 4 × 3 × 2 × 1
```

Notice how we've lined things up. You can see here that the 6! includes the 5!. In fact, 6! is just $6 \times 5!$. Let's continue:

Example: Factorials of consecutive numbers

```
Factorial of 4 = 4 × 3 × 2 × 1
Factorial of 3 =     3 × 2 × 1
Factorial of 2 =         2 × 1
Factorial of 1 =             1
```

[1] In mathematics, $n!$ normally means the factorial of a non-negative integer n, but that syntax is impossible in Haskell, so we don't use it here.

The factorial of any number is just that number multiplied by the factorial of the number one less than it. There's one exception: if we ask for the factorial of 0, we don't want to multiply 0 by the factorial of -1 (factorial is only for positive numbers). In fact, we just say the factorial of 0 is 1 (we *define* it to be so. Just take our word for it that this is right.[2]). So, 0 is the *base case* for the recursion: when we get to 0 we can immediately say that the answer is 1, no recursion needed. We can summarize the definition of the factorial function as follows:

- The factorial of 0 is 1.
- The factorial of any other number is that number multiplied by the factorial of the number one less than it.

We can translate this directly into Haskell:

Example: Factorial function

```
factorial 0 = 1
factorial n = n * factorial (n - 1)
```

This defines a new function called `factorial`. The first line says that the factorial of 0 is 1, and the second line says that the factorial of any other number `n` is equal to `n` times the factorial of `n - 1`. Note the parentheses around the `n - 1`; without them this would have been parsed as `(factorial n) - 1`; remember that function application (applying a function to a value) takes precedence over anything else when grouping isn't specified otherwise (we say that function application *binds more tightly* than anything else).

Note:
The `factorial` function above is best defined in a file, but since it is a small function, it is feasible to write it in GHCi as a one-liner. To do this, we need to add a semicolon to separate the lines:

```
    > let factorial 0 = 1; factorial n = n * factorial (n - 1)
```

Haskell actually uses line separation and other whitespace as a substitute for separation and grouping characters such as semicolons. Haskell programmers generally prefer the clean look of separate lines and appropriate indentation; still, explicit use of semicolons and other markers is always an alternative.

The example above demonstrate the simple relationship between factorial of a number, n, and the factorial of a slightly smaller number, n - 1.

Think of a function call as delegation. The instructions for a recursive function delegate a sub-task. It just so happens that the delegate function uses the same instructions as the delegator; it's only the input data that changes. The only really confusing thing about

[2] Actually, defining the factorial of 0 to be 1 is not just arbitrary; it's because the factorial of 0 represents an empty product ^{`https://en.wikipedia.org/wiki/empty%20product`} .

recursive functions is the fact that each function call uses the same parameter names, so it can be tricky to keep track of the many delegations.

Let's look at what happens when you execute `factorial 3`:

- 3 isn't 0, so we calculate the factorial of 2
  - 2 isn't 0, so we calculate the factorial of 1
    - 1 isn't 0, so we calculate the factorial of 0
      - 0 *is* 0, so we return 1.
    - To complete the calculation for factorial 1, we multiply the current number, 1, by the factorial of 0, which is 1, obtaining 1 (1 × 1).
  - To complete the calculation for factorial 2, we multiply the current number, 2, by the factorial of 1, which is 1, obtaining 2 (2 × 1 × 1).
- To complete the calculation for factorial 3, we multiply the current number, 3, by the factorial of 2, which is 2, obtaining 6 (3 × 2 × 1 × 1).

(Note that we end up with the one appearing twice, since the base case is 0 rather than 1; but that's okay since multiplying by 1 has no effect. We could have designed `factorial` to stop at 1 if we had wanted to, but the convention (which is often useful) is to define the factorial of 0.)

When reading or composing recursive functions, you'll rarely need to ”unwind” the recursion bit by bit — we leave that to the compiler.

One more note about our recursive definition of `factorial`: the order of the two declarations (one for `factorial 0` and one for `factorial n`) *is* important. Haskell decides which function definition to use by starting at the top and picking the first one that matches. If we had the general case (`factorial n`) before the 'base case' (`factorial 0`), then the general `n` would match *anything* passed into it – including 0. The compiler would then conclude that `factorial 0` equals `0 * factorial (-1)`, and so on to negative infinity (clearly not what we want). So, **always list multiple function definitions starting with the most specific and proceeding to the most general.**

Exercises:

1. Type the factorial function into a Haskell source file and load it into GHCi.
2. Try examples like `factorial 5` and `factorial 1000`.[a]
   - What about `factorial (-1)`? Why does this happen?
3. The *double factorial* of a number n is the product of *every other* number from 1 (or 2) up to n. For example, the double factorial of 8 is 8 × 6 × 4 × 2 = 384, and the double factorial of 7 is 7 × 5 × 3 × 1 = 105. Define a `doublefactorial` function in Haskell.

[a] Interestingly, older scientific calculators can't handle things like factorial of 1000 because they run out of memory with that many digits!

12.1.2 Loops, recursion, and accumulating parameters

Imperative languages use *loops* in the same sorts of contexts where Haskell programs use recursion. For example, an idiomatic way of writing a factorial function in C, a typical imperative language, would be using a *for* loop, like this:

Example: The factorial function in an imperative language

```
int factorial(int n) {
  int res = 1;
  for ( ; n > 1; n--)
    res *= n;
  return res;
}
```

Here, the for loop causes `res` to be multiplied by `n` repeatedly. After each repetition, `1` is subtracted from `n` (that is what `n--` does). The repetitions stop when `n` is no longer greater than `1`.

A straightforward translation of such a function to Haskell is not possible, since changing the value of the variables `res` and `n` (a destructive update) would not be allowed. However, you can always translate a loop into an equivalent recursive form by making each loop variable into an argument of a recursive function. For example, here is a recursive ”translation” of the above loop into Haskell:

Example: Using recursion to simulate a loop

```
factorial n = go n 1
    where
    go n res
        | n > 1     = go (n - 1) (res * n)
        | otherwise = res
```

`go` is an auxiliary function which actually performs the factorial calculation. It takes an extra argument, `res`, which is used as an *accumulating parameter* to build up the final result.

Note:
Depending on the languages you are familiar with, you might have concerns about performance problems caused by recursion. However, compilers for Haskell and other functional programming languages include a number of optimizations for recursion, (not surprising given how often recursion is needed). Also, Haskell is lazy — calculations are only performed once their results are required by other calculations, and that helps to avoid some of the performance problems. We'll discuss such issues and some of the subtleties they involve further in later chapters.

12.1.3 Other recursive functions

As it turns out, there is nothing particularly special about the `factorial` function; a great many numeric functions can be defined recursively in a natural way. For example, let's think about multiplication. When you were first learning multiplication (remember that moment? :)), it may have been through a process of 'repeated addition'. That is, 5 × 4 is the same as summing four copies of the number 5. Of course, summing four copies of 5 is the same as summing three copies, and then adding one more – that is, $5 \times 4 = 5 \times 3 + 5$. This leads us to a natural recursive definition of multiplication:

Example: Multiplication defined recursively

```
mult _ 0 = 0                       -- anything times 0 is zero
mult n 1 = n                       -- anything times 1 is itself
mult n m = (mult n (m - 1)) + n    -- recurse: multiply by one less, and add an
 extra copy
```

Stepping back a bit, we can see how numeric recursion fits into the general recursive pattern. The base case for numeric recursion usually consists of one or more specific numbers (often 0 or 1) for which the answer can be immediately given. The recursive case computes the result by calling the function recursively with a smaller argument and using the result in some manner to produce the final answer. The 'smaller argument' used is often one less than the current argument, leading to recursion which 'walks down the number line' (like the examples of `factorial` and `mult` above). However, the prototypical pattern is not the only possibility; the smaller argument could be produced in some other way as well.

Exercises:

1. Expand out the multiplication 5 × 4 similarly to the expansion we used above for `factorial 3`.
2. Define a recursive function `power` such that `power x y` raises `x` to the `y` power.
3. You are given a function `plusOne x = x + 1`. Without using any other `(+)`s, define a recursive function `addition` such that `addition x y` adds `x` and `y` together.
4. (Harder) Implement the function `log2`, which computes the integer log (base 2) of its argument. That is, `log2` computes the exponent of the largest power of 2 which is less than or equal to its argument. For example, `log2 16 = 4`, `log2 11 = 3`, and `log2 1 = 0`. (Small hint: read the last phrase of the paragraph immediately preceding these exercises.)

12.2 List-based recursion

Haskell has many recursive functions, especially concerning lists.[3] Consider the `length` function that finds the length of a list:

Example: The recursive definition of `length`

```
length :: [a] -> Int
length []     = 0
length (x:xs) = 1 + length xs
```

So, the type signature of `length` tells us that it takes any type of list and produces an `Int`. The next line says that the length of an empty list is 0 (this is the base case). The final line is the recursive case: if a list isn't empty, then it can be broken down into a first element (here called `x`) and the rest of the list (which will just be the empty list if there are no more elements) which will, by convention, be called `xs` (i.e. plural of `x`). The length of the list is 1 (accounting for the `x`) plus the length of `xs` (as in the `tail` example in Next steps[4], `xs` is set when the argument list matches the (:) pattern).

Consider the concatenation function `(++)` which joins two lists together:

Example: The recursive `(++)`

```
  Prelude> [1,2,3] ++ [4,5,6]
  [1,2,3,4,5,6]
  Prelude> "Hello " ++ "world" -- Strings are lists of Chars
  "Hello world"
```

```
(++) :: [a] -> [a] -> [a]
[] ++ ys     = ys
(x:xs) ++ ys = x : xs ++ ys
```

This is a little more complicated than `length`. The type says that `(++)` takes two lists of the same type and produces another list of the same type. The base case says that concatenating the empty list with a list `ys` is the same as `ys` itself. Finally, the recursive case breaks the first list into its head (`x`) and tail (`xs`) and says that to concatenate the two lists, concatenate the tail of the first list with the second list, and then tack the head `x` on the front.

[3] This is no coincidence; without mutable variables, recursion is the only way to implement control structures. This might sound like a limitation until you get used to it.

[4] Chapter 9 on page 49

There's a pattern here: with list-based functions, the base case usually involves an empty list, and the recursive case involves passing the tail of the list to our function again, so that the list becomes progressively smaller.

Exercises:
Give recursive definitions for the following list-based functions. In each case, think what the base case would be, then think what the general case would look like, in terms of everything smaller than it. (Note that all of these functions are available in Prelude, so you will want to give them different names when testing your definitions in GHCi.)

1. `replicate :: Int -> a -> [a]`, which takes a count and an element and returns the list which is that element repeated that many times. E.g. `replicate 3 'a' = "aaa"`. (Hint: think about what replicate of anything with a count of 0 should be; a count of 0 is your 'base case'.)
2. `(!!) :: [a] -> Int -> a`, which returns the element at the given 'index'. The first element is at index 0, the second at index 1, and so on. Note that with this function, you're recursing *both* numerically and down a list[a].
3. (A bit harder.) `zip :: [a] -> [b] -> [(a, b)]`, which takes two lists and 'zips' them together, so that the first pair in the resulting list is the first two elements of the two lists, and so on. E.g. `zip [1,2,3] "abc" = [(1, 'a'), (2, 'b'), (3, 'c')]`. If either of the lists is shorter than the other, you can stop once either list runs out. E.g. `zip [1,2] "abc" = [(1, 'a'), (2, 'b')]`.
4. Define `length` using an auxiliary function and an accumulating parameter, as in the loop-like alternate version of `factorial`.

[a] Incidentally, `(!!)` provides a reasonable solution for the problem of the fourth exercise in Lists and tuples/Retrieving values ^{`https://en.wikibooks.org/wiki/..%2FLists%20and%20tuples%23Pending%20questions`} .

Recursion is used to define nearly all functions to do with lists and numbers. The next time you need a list-based algorithm, start with a case for the empty list and a case for the non-empty list and see if your algorithm is recursive.

12.3 Don't get TOO excited about recursion...

Despite its ubiquity in Haskell, one rarely has to write functions that are explicitly recursive. Instead, standard library functions perform recursion for us in various ways. For example, a simpler way to implement the `factorial` function is:

Example: Implementing factorial with a standard library function

```
factorial n = product [1..n]
```

Almost seems like cheating, doesn't it? :) This is the version of `factorial` that most experienced Haskell programmers would write, rather than the explicitly recursive version we started out with. Of course, the `product` function uses some list recursion behind the scenes,[5] but writing `factorial` in this way means you, the programmer, don't have to worry about it.

[5] Actually, it uses a function called `foldl`, which actually does the recursion.

13 Lists II

Earlier, we learned that Haskell builds lists via the cons operator `(:)` and the empty list `[]`. We saw how we can work on lists bit by bit using a combination of recursion and pattern matching. In this chapter and the next, we will consider more in-depth techniques for list processing and discover some new notation. We will get our first taste of Haskell features like infinite lists, list comprehensions, and higher-order functions.

> **Note:**
> Throughout this chapter, you will read and write functions which sum, subtract, and multiply elements of lists. For simplicity's sake, we will pretend that list elements are of type `Integer`. However, as you will recall from the discussions on Type basics II[a], there are many different types with the Num typeclass. As an exercise of sorts, you could figure out what the type signatures of such functions would be if we made them polymorphic, allowing for the list elements to have any type in the class `Num`. To check your signatures, just omit them temporarily, load the functions into GHCi, use :t and let type inference guide you.

[a] Chapter 7 on page 37

13.1 Rebuilding lists

Here's a function that doubles every element from a list of integers:

```
doubleList :: [Integer] -> [Integer]
doubleList [] = []
doubleList (n:ns) = (2 * n) : doubleList ns
```

Here, the base case is the empty list which evaluates to an empty list. In the recursive case, doubleList *builds up a new list* by using `(:)`. The first element of this new list is twice the head of the argument, and we obtain the rest of the result by recursively calling doubleList on the tail of the argument. When the tail gets to an empty list, the base case will be invoked and recursion will stop.[1]

Let's study the evaluation of an example expression:

```
doubleList [1,2,3,4]
```

We can work it out longhand by substituting the argument into the function definition, just like schoolbook algebra:

[1] Had we forgotten the base case, once the recursion got to an empty list the (x:xs) pattern match would fail, and we would get an error.

```
doubleList 1:[2,3,4] = (1*2) : doubleList (2 : [3,4])
                     = (1*2) : (2*2) : doubleList (3 : [4])
                     = (1*2) : (2*2) : (3*2) : doubleList (4 : [])
                     = (1*2) : (2*2) : (3*2) : (4*2) : doubleList []
                     = (1*2) : (2*2) : (3*2) : (4*2) : []
                     = 2 : 4 : 6 : 8 : []
                     = [2, 4, 6, 8]
```

Thus, we rebuilt the original list replacing every element by its double.

In this longhand evaluation exercise, the *moment* at which we choose to evaluate the multiplications does not affect the result. We could just as well have evaluated the doublings immediately after each recursive call of doubleList.[2]

Haskell uses this flexibility on evaluation order in some important ways. As a *pure* functional programming language, the compiler makes most of the decisions about when to actually evaluate things. As a *lazy* language, Haskell usually defers evaluation until a final value is needed (which may sometimes never occur).[3] From the programmer's point of view, evaluation order rarely matters.[4]

13.1.1 Generalizing

To triple each element in a list, we could follow the same strategy as with `doubleList`:

```
tripleList :: [Integer] -> [Integer]
tripleList [] = []
tripleList (n:ns) = (3 * n) : tripleList ns
```

But we don't want to write a new list-multiplying function for every different multiplier (such as multiplying the elements of a list by 4, 8, 17 etc.). So, let's make a general function to allow multiplication by *any* number. Our new function will take two arguments: the multiplicand as well as a list of `Integer`s to multiply:

```
multiplyList :: Integer -> [Integer] -> [Integer]
multiplyList _ [] = []
multiplyList m (n:ns) = (m * n) : multiplyList m ns
```

This example deploys `_` as a ”don't care” pattern. The multiplicand is not used for the base case, so we ignore that argument instead of giving it a name (like `m`, `n`, or `ns`).

We can test `multiplyList` to see that it works as expected:

```
Prelude> multiplyList 17 [1,2,3,4]
[17,34,51,68]
```

[2] ...as long as none of the calculations result in an error or nontermination, which are not problems in this case.

[3] The compiler may sometimes evaluate things sooner in order to improve efficiency.

[4] One exception is the case of *infinite lists* (!) which we will consider in a short while.

Exercises:
Write the following functions and test them out. Don't forget the type signatures.

1. takeInt returns the first *n* items in a list. So, `takeInt 4 [11,21,31,41,51,61]` returns `[11,21,31,41]`.
2. dropInt drops the first *n* items in a list and returns the rest. So, `dropInt 3 [11,21,31,41,51]` returns `[41,51]`.
3. sumInt returns the sum of the items in a list.
4. scanSum adds the items in a list and returns a list of the running totals. So, `scanSum [2,3,4,5]` returns `[2,5,9,14]`.
5. diffs returns a list of the differences between adjacent items. So, `diffs [3,5,6,8]` returns `[2,1,2]`. (Hints: one solution involves writing an auxiliary function which takes two lists and calculates the difference between corresponding elements. Alternatively, you might explore the fact that lists with at least two elements can be matched to a `(x:y:ys)` pattern.) The first three functions are in Prelude under the names `take`, `drop`, and `sum`.

13.2 Generalizing even further

In this chapter, we started with a function constrained to multiplying the elements by `2`. Then, we recognized that we could avoid hard-coding a new function for each multiplicand by making `multiplyList` to easily use any `Integer`. Now, what if we wanted a different operator such as addition or to compute the square of each element?

We can generalize still further using a key functionality of Haskell. However, because the solution can seem surprising, we will approach it in a somewhat roundabout way. Consider the type signature of `multiplyList`:

```
multiplyList :: Integer -> [Integer] -> [Integer]
```

The first thing to know is that the `->` arrow in type signatures is *right associative.* That means we can read this signature as:

```
multiplyList :: Integer -> ([Integer] -> [Integer])
```

How should we understand that? It tells us that `multiplyList` is a function that takes *one* `Integer` argument and evaluates to *another function.* The function it returns happens to take a list of `Integer`s and return another list of `Integer`s.

Consider our old `doubleList` function redefined in terms of `multiplyList`:

```
doubleList :: [Integer] -> [Integer]
doubleList xs = multiplyList 2 xs
```

Writing this way, we can clearly cancel out the ‘xs‘:

```
doubleList = multiplyList 2
```

This definition style (with no argument variables) is called *point-free* style. Our definition now says that applying only one argument to `multiplyList` doesn't fail to evaluate, rather it gives us a more specific function of type `[Integer] -> [Integer]` instead of finishing with a final `[Integer]` value.

We now see that functions in Haskell behave much like any other value. Functions can return other functions, and functions can stand alone as objects without mentioning their arguments. Functions seem almost like normal constants. Can we use functions themselves as *arguments* even? Yes, and that's the key to the next step in generalizing `multiplyList`. We need a function that takes any other appropriate function and applies the given function to the elements of a list:

```
applyToIntegers :: (Integer -> Integer) -> [Integer] -> [Integer]
applyToIntegers _ [] = []
applyToIntegers f (n:ns) = (f n) : applyToIntegers f ns
```

With `applyToIntegers`, we can take *any* `Integer -> Integer` function and apply it to the elements of a list of `Integers`. We can thus use this generalized function to redefine `multiplyList`:

```
multiplyList :: Integer -> [Integer] -> [Integer]
multiplyList m = applyToIntegers ((*) m)
```

That uses the `(*)` function with a single initial argument to create a new function which is ready to take one more argument (which, in this use case, will come from the numbers in a given list).

13.2.1 Currying

If all this abstraction confuses you, consider a concrete example: When we multiply 5 * 7 in Haskell, the `(*)` function doesn't just take two arguments at once, it actually first takes the 5 and returns a new *5** function; and that new function *then* takes a second argument and multiplies that by 5. So, for our example, we then give the 7 as an argument to the 5* function, and *that* returns us our final evaluated number (35).

So, **all functions in Haskell really take only one argument**. However, when we know how many intermediate functions we will generate to reach a final result, we can *treat* functions *as if* they take many arguments. The number of arguments we generally talk about functions taking is actually the number of one-argument functions we get between the first argument and a final, non-functional result value.

The process of creating intermediate functions when feeding arguments into a complex function is called *currying* (named after Haskell Curry, also the namesake of the Haskell programming language).

13.3 The `map` function

While `applyToIntegers` has type `(Integer -> Integer) -> [Integer] -> [Integer]`, the definition itself contains nothing specific to integers. To use the same logic with other types of lists, we could define versions such as `applyToChars`, `applyToStrings` and so on. They would all have the same definition but different type signatures. We can avoid all that redundancy with the final step in generalizing: making a fully polymorphic version with signature `(a -> b) -> [a] -> [b]`. Prelude already has this function, and it is called `map`:

```
map :: (a -> b) -> [a] -> [b]
map _ [] = []
map f (x:xs) = (f x) : map f xs
```

With `map`, we can effortlessly implement functions as different as...

```
multiplyList :: Integer -> [Integer] -> [Integer]
multiplyList m = map ((*) m)
```

... and...

```
heads :: [[a]] -> [a]
heads = map head
```

```
Prelude> heads [[1,2,3,4],[4,3,2,1],[5,10,15]]
[1,4,5]
```

`map` is the general solution for applying a function to each and every element of a list. Our original `doubleList` problem was simply a specific version of `map`. Functions like `map` which take other functions as arguments are called *higher-order functions*. We will learn about more higher-order functions for list processing in the next chapter.

Exercises:

1. Use `map` to build functions that, given a list xs of Ints, return:
   - A list that is the element-wise negation of xs.
   - A list of lists of Ints xss that, for each element of xs, contains the divisors of xs. You can use the following function to get the divisors:
     `divisors p = [ f | f <- [1..p], p `mod` f == 0 ]`
   - The element-wise negation of xss.
2. Implement a Run Length Encoding (RLE) encoder and decoder.
   - The idea of RLE is simple; given some input:
     `"aaaabbaaa"`
     compress it by taking the length of each run of characters:`(4,'a'), (2, 'b'), (3, 'a')`
   - The `concat` and `group` functions might be helpful. In order to use `group`, import the Data.List module by typing `:m Data.List` at the ghci prompt or by adding `import Data.List` to your Haskell source code file.
   - What is the type of your `encode` and `decode` functions?
   - How would you convert the list of tuples (e.g. `[(4,'a'), (6,'b')]`) into a string (e.g. ”4a6b”)?
   - (bonus) Assuming numeric characters are forbidden in the original string, how would you parse that string back into a list of tuples?

13.4 Tips and Tricks

A few miscellaneous notes about lists in Haskell:

13.4.1 Dot Dot Notation

Haskell has a convenient shorthand for writing ordered lists of regularly-spaced integers. Some examples to illustrate it:

```
Code            Result
----            ------
[1..10]         [1,2,3,4,5,6,7,8,9,10]
[2,4..10]       [2,4,6,8,10]
[5,4..1]        [5,4,3,2,1]
[1,3..10]       [1,3,5,7,9]
```

The same notation works with characters and even with floating point numbers. Unfortunately, floating-point numbers are problematic due to rounding errors. Try this:

```
[0,0.1 .. 1]
```

Note:
The `..` notation *only* works with sequences with fixed differences between consecutive elements. For instance, you cannot write...

```
[0,1,1,2,3,5,8..100]
```

... and expect to magically get back the rest of the Fibonacci series.[a]

[a] http://en.wikipedia.org/wiki/Fibonacci_number

13.4.2 Infinite Lists

Thanks to lazy evaluation, Haskell lists can be *infinite*. For example, the following generates the infinite list of integers starting with 1:

```
[1..]
```

(If you try this in GHCi, remember you can stop an evaluation with Ctrl-c).

The same effect could be achieved with a recursive function:

```
intsFrom n = n : intsFrom (n + 1) -- note there is no base case!
positiveInts = intsFrom 1
```

Infinite lists are useful in practice because Haskell's lazy evaluation never actually evaluates more than it needs at any given moment. In most cases, we can treat an infinite list like an ordinary one. The program will only go into an infinite loop when evaluation requires all the values in the list. So, we can't sort or print an infinite list, but:

```
evens = doubleList [1..]
```

will define ”evens” to be the infinite list [2,4,6,8..], and we can *then* pass ”evens” into other functions that only need to evaluate part of the list for their *final* result. Haskell will know to only use the portion of the infinite list needed in the end.

Compared to hard-coding a long finite list, it's often more convenient to define an infinite list and then take the first few items. An infinite list can also be a handy alternative to the traditional endless loop at the top level of an interactive program.

13.4.3 A note about `head` and `tail`

Given the choice of using either the `( : )` pattern or `head`/`tail` to split lists, pattern matching is almost always preferable. It may be tempting to use `head` and `tail` due to simplicity and terseness, but it is too easy to forget that they fail on empty lists (and runtime crashes are never good). We do have a Prelude function, `null :: [a] -> Bool`, which returns `True` for empty lists and `False` otherwise, so that provides a sane way of checking for empty lists without pattern matching; but matching an empty list tends to be cleaner and clearer than the corresponding if-then-else expression using `null`.

Exercises:

1. With respect to your solutions to the first set of exercises in this chapter, is there any difference between `scanSum (takeInt 10 [1..])` and `takeInt 10 (scanSum [1..])`?
2. Write functions that, when applied to lists, give the *last* element of the list and the list with the last element dropped.
This functionality is provided by Prelude through the `last` and `init` functions. Like `head` and `tail`, they blow up when given empty lists.

14 Lists III

14.1 Folds

Like `map`, a *fold* is a higher order function that takes a function and a list. However, instead of applying the function element by element, the fold uses it to combine the list elements into a result value.

Let’s look at a few concrete examples. `sum` could be implemented as:

Example: sum

```
sum :: [Integer] -> Integer
sum []     = 0
sum (x:xs) = x + sum xs
```

and `product` as:

Example: product

```
product :: [Integer] -> Integer
product []     = 1
product (x:xs) = x * product xs
```

`concat`, which takes a list of lists and joins (concatenates) them into one:

Example: concat

```
concat :: [[a]] -> [a]
concat []     = []
concat (x:xs) = x ++ concat xs
```

All these examples show a pattern of recursion known as a fold. Think of the name referring to a list getting ”folded up” into a single value or to a function being ”folded between” the elements of the list.

Prelude defines four `fold` functions: `foldr`, `foldl`, `foldr1` and `foldl1`.

14.1.1 foldr

The *right-associative* `foldr` folds up a list from the right to left. As it proceeds, foldr uses the given function to combine each of the elements with the running value called the *accumulator*. When calling foldr, the initial value of the accumulator is set as an argument.

```
foldr            :: (a -> b -> b) -> b -> [a] -> b
foldr f acc []     = acc
foldr f acc (x:xs) = f x (foldr f acc xs)
```

The first argument to `foldr` is a function with two arguments. The second argument is value for the accumulator (which often starts at a neutral "zero" value). The third argument is the list to be folded.

In `sum`, `f` is `(+)`, and `acc` is `0`. In `concat`, `f` is `(++)` and `acc` is `[]`. In many cases (like all of our examples so far), the function passed to a fold will be one that takes two arguments of the same type, but this is not necessarily the case (as we can see from the `(a -> b -> b)` part of the type signature — if the types *had* to be the same, the first two letters in the type signature would have matched).

Remember, a list in Haskell written as `[a, b, c]` is an alternative (syntactic sugar) style for `a : b : c : []`.

Now, `foldr f acc xs` in the `foldr` definition simply replaces each cons (:) in the `xs` list with the function `f` while replacing the empty list at the end with `acc`:

```
foldr f acc (a:b:c:[]) = f a (f b (f c acc))
```

Note how the parentheses nest around the right end of the list.

An elegant visualisation is given by picturing the list data structure as a tree:

```
    :                         f
   / \                       / \
  a   :       foldr f acc   a   f
     / \    ------------->     / \
    b   :                     b   f
       / \                       / \
      c  []                     c   acc
```

We can see here that `foldr (:) []` will return the list completely unchanged. That sort of function that has no effect is called an *identity function*. You should start building a habit of looking for identity functions in different cases, and we'll discuss them more later when we learn about *monoids*.

14.1.2 foldl

The *left-associative* `foldl` processes the list in the opposite direction, starting at the left side with the first element.

```
foldl            :: (a -> b -> a) -> a -> [b] -> a
foldl f acc []     =  acc
foldl f acc (x:xs) =  foldl f (f acc x) xs
```

So, brackets in the resulting expression accumulate around the left end of the list:

```
foldl f acc (a:b:c:[]) = f (f (f acc a) b) c
```

The corresponding trees look like:

```
  :                            f
 / \                          / \
a   :       foldl f acc      f   c
   / \    ------------->    / \
  b   :                    f   b
     / \                  / \
    c  []               acc a
```

Because all folds include both left and right elements, beginners can get confused by the names. You could think of foldr as short for fold-right-to-left and foldl as fold-left-to-right. The names refer to where the fold starts.

Note:
Technical Note: foldl is *tail-recursive*, that is, it recurses immediately, calling itself. For this reason the compiler will optimise it to a simple loop for efficiency. However, Haskell is a lazy language, so the calls to *f* will be left unevaluated by default, thus building up an unevaluated expression in memory that includes the entire length of the list. To avoid running out of memory, we have a version of foldl called `foldl'`that is *strict* — it forces the evaluation of *f* immediately at each step.
An apostrophe at the end of a function name is pronounced ”tick” as in ”fold-L-tick”. A tick is a valid character in Haskell identifiers. `foldl'` can be found in the `Data.List` library module (imported by adding `import Data.List` to the beginning of a source file). As a rule of thumb, you should use `foldr` on lists that might be infinite or where the fold is building up a data structure and use `foldl'` if the list is known to be finite and comes down to a single value. There is almost never a good reason to use `foldl` (without the tick), though it might just work if the lists fed to it are not too long.

14.1.3 foldr1 and foldl1

As previously noted, the type declaration for `foldr` makes it quite possible for the list elements and result to be of different types. For example, ”read” is a function that takes a string and converts it into some type (the type system is smart enough to figure out which one). In this case we convert it into a float.

Example: The list elements and results can have different types

```
addStr :: String -> Float -> Float
addStr str x = read str + x

sumStr :: [String] -> Float
sumStr = foldr addStr 0.0
```

There is also a variant called `foldr1` ("fold - R - one") which dispenses with an explicit "zero" for an accumulator by taking the last element of the list instead:

```
foldr1           :: (a -> a -> a) -> [a] -> a
foldr1 f [x]     =  x
foldr1 f (x:xs)  =  f x (foldr1 f xs)
foldr1 _ []      =  error "Prelude.foldr1: empty list"
```

And `foldl1` as well:

```
foldl1           :: (a -> a -> a) -> [a] -> a
foldl1 f (x:xs)  =  foldl f x xs
foldl1 _ []      =  error "Prelude.foldl1: empty list"
```

Note: Just like for foldl, the Data.List library includes foldl1' as a strict version of foldl1.

With foldl1 and foldr1, all the types have to be the same, and an empty list is an error. These variants are useful when there is no obvious candidate for the initial accumulator value and we are sure that the list is not going to be empty. When in doubt, stick with foldr or foldl'.

14.1.4 folds and laziness

One reason that right-associative folds are more natural in Haskell than left-associative ones is that right folds can operate on infinite lists. A fold that returns an infinite list is perfectly usable in a larger context that doesn't need to access the entire infinite result. In that case, foldr can move along as much as needed and the compiler will know when to stop. However, a left fold necessarily calls itself recursively until it reaches the end of the input list (because the recursive call is not made in an argument to f). Needless to say, no end will be reached if an input list to foldl is infinite.

As a toy example, consider a function `echoes` that takes a list of integers and produces a list such that wherever the number n occurs in the input list, it is replicated n times in the output list. To create our echoes function, we will use the prelude function `replicate` in which `replicate n x` is a list of length n with x the value of every element.

We can write echoes as a foldr quite handily:

```
echoes = foldr (\ x xs -> (replicate x x) ++ xs) []
take 10 (echoes [1..])     -- [1,2,2,3,3,3,4,4,4,4]
```

(*Note:* This definition is compact thanks to the `\ x xs ->`syntax. The `\`, meant to look like a lambda (λ), works as an *unnamed* function for cases where we won't use the function again anywhere else. Thus, we provide the definition of our one-time function *in situ.* In this case, `x` and `xs` are the arguments, and the right-hand side of the definition is what comes after the `->`.)

We could have instead used a foldl:

```
echoes = foldl (\xs x -> xs ++ (replicate x x)) []
take 10 (echoes [1..])     -- not terminating
```

but only the foldr version works on an infinite lists. What would happen if you just evaluate `echoes [1..]`? Try it! (If you try this in GHCi or a terminal, remember you can stop an evaluation with Ctrl-c, but you have to be quick and keep an eye on the system monitor or your memory will be consumed in no time and your system will hang.)

As a final example, `map` itself can be implemented as a fold:

```
map f = foldr (\x xs -> f x : xs) []
```

Folding takes some time to get used to, but it is a fundamental pattern in functional programming and eventually becomes very natural. Any time you want to traverse a list and build up a result from its members, you likely want a fold.

Exercises:

1. Define the following functions recursively (like the definitions for `sum`, `product` and `concat` above), then turn them into a fold:
   - `and :: [Bool] -> Bool`, which returns True if a list of Bools are all True, and False otherwise.
   - `or :: [Bool] -> Bool`, which returns True if any of a list of Bools are True, and False otherwise.
2. Define the following functions using `foldl1` or `foldr1`:
   - `maximum :: Ord a => [a] -> a`, which returns the maximum element of a list (hint: `max :: Ord a => a -> a -> a` returns the maximum of two values).
   - `minimum :: Ord a => [a] -> a`, which returns the minimum element of a list (hint: `min :: Ord a => a -> a -> a` returns the minimum of two values).
3. Use a fold (*which one?*) to define `reverse :: [a] -> [a]`, which returns a list with the elements in reverse order.

Note that all of these are Prelude functions, so they will be always close at hand when you need them. (Also, that means you must use slightly different names in order to test your answers in GHCi.)

14.2 Scans

A ”scan” is like a cross between a `map` and a fold. Folding a list accumulates a single return value, whereas mapping puts each item through a function returning a separate result for each item. A scan does both: it accumulates a value like a fold, but instead of returning only a final value it returns a list of all the intermediate values.

Prelude contains four scan functions:

```
scanl   :: (a -> b -> a) -> a -> [b] -> [a]
```

`scanl` accumulates the list from the left, and the second argument becomes the first item in the resulting list. So, `scanl (+) 0 [1,2,3] = [0,1,3,6]`.

```
scanl1  :: (a -> a -> a) -> [a] -> [a]
```

`scanl1` uses the first item of the list as a zero parameter. It is what you would typically use if the input and output items are the same type. Notice the difference in the type signatures between `scanl` and `scanl1`. `scanl1 (+) [1,2,3] = [1,3,6]`.

```
scanr   :: (a -> b -> b) -> b -> [a] -> [b]
scanr (+) 0 [1,2,3] = [6,5,3,0]
scanr1  :: (a -> a -> a) -> [a] -> [a]
scanr1 (+) [1,2,3] = [6,5,3]
```

These two functions are the counterparts of `scanl` and `scanl1` that accumulate the totals from the right.

Exercises:

1. Write your own definition of `scanr`, first using recursion, and then using `foldr`. Do the same for `scanl` first using recursion then `foldl`.
2. Define the following functions:
   - `factList :: Integer -> [Integer]`, which returns a list of factorials from 1 up to its argument. For example, `factList 4 = [1,2,6,24]`. *More to be added*

14.3 filter

A common operation performed on lists is filtering[1] — generating a new list composed only of elements of the first list that meet a certain condition. A simple example: making a list of only even numbers from a list of integers.

```
retainEven :: [Int] -> [Int]
retainEven [] = []
retainEven (n:ns) =
-- mod n 2 computes the remainder for the integer division of n by 2.
  if (mod n 2) == 0
    then n : (retainEven ns)
    else retainEven ns
```

This definition is somewhat verbose and specific. Prelude provides a concise and general `filter` function with type signature:

```
filter :: (a -> Bool) -> [a] -> [a]
```

[1] https://en.wikipedia.org/wiki/Filter%20%28mathematics%29

So, a `(a -> Bool)` function tests an elements for some condition, we then feed in a list to be filtered, and we get back the filtered list.

To write `retainEven` using `filter`, we need to state the condition as an auxiliary `(a -> Bool)` function, like this one:

```
isEven :: Int -> Bool
isEven n = (mod n 2) == 0
```

And then retainEven becomes simply:

```
retainEven ns = filter isEven ns
```

We used ns instead of xs to indicate that we know these are numbers and not just anything, but we can ignore that and use a more terse point-free definition:

```
retainEven = filter isEven
```

This is like what we demonstrated before for `map` and the folds. Like `filter`, those take another function as argument; and using them point-free emphasizes this "functions-of-functions" aspect.

14.4 List comprehensions

List comprehensions are syntactic sugar for some common list operations, such as filtering. For instance, instead of using the Prelude `filter`, we could write `retainEven` like this:

```
retainEven es = [n | n <- es, isEven n]
```

This compact syntax may look intimidating, but it is simple to break down. One interpretation is:

- (Starting from the middle) Take the list *es* and draw (the "<-") each of its elements as a value *n*.
- (After the comma) For each drawn *n* test the boolean condition `isEven n`.
- (Before the vertical bar) If (and only if) the boolean condition is satisfied, append *n* to the new list being created (note the square brackets around the whole expression).

Thus, if `es` is [1,2,3,4], then we would get back the list [2,4]. 1 and 3 were not drawn because `(isEven n) == False`.

The power of list comprehensions comes from being easily extensible. Firstly, we can use as many tests as we wish (even zero!). Multiple conditions are written as a comma-separated list of expressions (which should evaluate to a Boolean, of course). For a simple example, suppose we want to modify `retainEven` so that only numbers larger than 100 are retained:

```
retainLargeEvens :: [Int] -> [Int]
retainLargeEvens es = [n | n <- es, isEven n, n > 100]
```

Furthermore, we are not limited to using `n` as the element to be appended when generating a new list. Instead, we could place any expression before the vertical bar (if it is compatible with the type of the list, of course). For instance, if we wanted to subtract one from every even number, all it would take is:

```
evensMinusOne es = [n - 1 | n <- es, isEven n]
```

In effect, that means the list comprehension syntax incorporates the functionalities of `map` and `filter`.

To further sweeten things, the left arrow notation in list comprehensions can be combined with pattern matching. For example, suppose we had a list of `(Int, Int)` tuples, and we would like to construct a list with the first element of every tuple whose second element is even. Using list comprehensions, we might write it as follows:

```
firstForEvenSeconds :: [(Int, Int)] -> [Int]
firstForEvenSeconds ps = [fst p | p <- ps, isEven (snd p)] -- here, p is for
 pairs.
```

Patterns can make it much more readable:

```
firstForEvenSeconds ps = [x | (x, y) <- ps, isEven y]
```

As in other cases, arbitrary expressions may be used before the `|`. If we wanted a list with the double of those first elements:

```
doubleOfFirstForEvenSeconds :: [(Int, Int)] -> [Int]
doubleOfFirstForEvenSeconds ps = [2 * x | (x, y) <- ps, isEven y]
```

Not counting spaces, that function code is shorter than its descriptive name!

There are even more possible tricks:

```
allPairs :: [(Int, Int)]
allPairs = [(x, y) | x <- [1..4], y <- [5..8]]
```

This comprehension draws from *two* lists, and generates all possible `(x, y)` pairs with the first element drawn from `[1..4]` and the second from `[5..8]`. In the final list of pairs, the first elements will be those generated with the first element of the first list (here, `1`), then those with the second element of the first list, and so on. In this example, the full list is (linebreaks added for clarity):

```
Prelude> [(x, y) | x <- [1..4], y <- [5..8]]
[(1,5),(1,6),(1,7),(1,8),
 (2,5),(2,6),(2,7),(2,8),
 (3,5),(3,6),(3,7),(3,8),
 (4,5),(4,6),(4,7),(4,8)]
```

We could easily add a condition to restrict the combinations that go into the final list:

```
somePairs = [(x, y) | x <- [1..4], y <- [5..8], x + y > 8]
```

This list only has the pairs with the sum of elements larger than 8; starting with `(1,8)`, then `(2,7)` and so forth.

Exercises:

1. Write a `returnDivisible :: Int -> [Int] -> [Int]` function which filters a list of integers retaining only the numbers divisible by the integer passed as first argument. For integers x and n, x is divisible by n if `(mod x n) == 0` (note that the test for evenness is a specific case of that).
2.
3. Write a function `choosingTails :: [[Int]] -> [[Int]]` using list comprehension syntax with appropriate guards (filters) for empty lists returning a list of tails following a head bigger than 5:

   ```
   choosingTails  [[7,6,3],[],[6,4,2],[9,4,3],[5,5,5]]
   -- [[6,3],[4,2],[4,3]]
   ```
4.
5. Does the order of guards matter? You may find it out by playing with the function of the preceding exercise.
6.
7. Over this section we've seen how list comprehensions are essentially syntactic sugar for `filter` and `map`. Now work in the opposite direction and define alternative versions of the `filter` and `map` using the list comprehension syntax.
8.
9. Rewrite `doubleOfFirstForEvenSeconds` using `filter` and `map` instead of list comprehension.

15 Type declarations

You're not restricted to working with just the types provided by default with the language. There are many benefits to defining your own types:

- Code can be written in terms of the problem being solved, making programs easier to design, write and understand.
- Related pieces of data can be brought together in ways more convenient and meaningful than simply putting and getting values from lists or tuples.
- Pattern matching and the type system can be used to their fullest extent by making them work with your custom types.

Haskell has three basic ways to declare a new type:

- The `data` declaration, which defines new data types.
- The `type` declaration for type synonyms, that is, alternative names for existing types.
- The `newtype` declaration, which defines new data types equivalent to existing ones.

In this chapter, we will study `data` and `type`. In a later chapter, we will discuss `newtype` and see where it can be useful.

15.1 `data` and constructor functions

`data` is used to define new data types mostly using existing ones as building blocks. Here's a data structure for elements in a simple list of anniversaries:

```
data Anniversary = Birthday String Int Int Int       -- name, year, month, day
                 | Wedding String String Int Int Int -- spouse name 1, spouse
 name 2, year, month, day
```

This declares a new data type `Anniversary`, which can be either a Birthday or a Wedding. A Birthday *contains* one string and three integers and a Wedding *contains* two strings and three integers. The definitions of the two possibilities are separated by the vertical bar. The comments explain to readers of the code about the intended use of these new types. Moreover, with the declaration we also get two *constructor functions* for `Anniversary`; appropriately enough, they are called `Birthday` and `Wedding`. These functions provide a way to build a new `Anniversary`.

Types defined by `data` declarations are often referred to as *algebraic data types*, which is something we will address further in later chapters.

As usual with Haskell, the case of the first letter is important: type names and constructor functions must start with capital letters. Other than this syntactic detail, constructor functions work pretty much like the ”conventional” functions we have met so far. In fact, if you use `:t` in GHCi to query the type of, say, `Birthday`, you'll get:

- `Main> :t Birthday Birthday :: String -> Int -> Int -> Int -> Anniversary`

Meaning it's just a function which takes one String and three Int as arguments and *evaluates to* an `Anniversary`. This anniversary will contain the four arguments we passed as specified by the `Birthday` constructor.

Calling constructors is no different from calling other functions. For example, suppose we have John Smith born on 3rd July 1968:

```
johnSmith :: Anniversary
johnSmith = Birthday "John Smith" 1968 7 3
```

He married Jane Smith on 4th March 1987:

```
smithWedding :: Anniversary
smithWedding = Wedding "John Smith" "Jane Smith" 1987 3 4
```

These two anniversaries can, for instance, be put in a list:

```
anniversariesOfJohnSmith :: [Anniversary]
anniversariesOfJohnSmith = [johnSmith, smithWedding]
```

Or you could just as easily have called the constructors straight away when building the list (although the resulting code looks a bit cluttered).

```
anniversariesOfJohnSmith = [Birthday "John Smith" 1968 7 3, Wedding "John Smith"
 "Jane Smith" 1987 3 4]
```

15.2 Deconstructing types

To use our new data types, we must have a way to access their contents. For instance, one very basic operation with the anniversaries defined above would be extracting the names and dates they contain as a String. So we need a `showAnniversary` function (for the sake of code clarity, we used an auxiliary `showDate` function but let's ignore it for a moment):

```
showDate :: Int -> Int -> Int -> String
showDate y m d = show y ++ "-" ++ show m ++ "-" ++ show d

showAnniversary :: Anniversary -> String

showAnniversary (Birthday name year month day) =
   name ++ " born " ++ showDate year month day

showAnniversary (Wedding name1 name2 year month day) =
   name1 ++ " married " ++ name2 ++ " on " ++ showDate year month day
```

This example shows how we can deconstruct the values built in our data types. `showAnniversary` takes a single argument of type `Anniversary`. Instead of just providing a name for the argument on the left side of the definition, however, we specify one of the constructor functions and give names to each argument of the constructor (which correspond to the contents of the Anniversary). A more formal way of describing this ”giving names” process is to say we are *binding variables*. ”Binding” is being used in the sense of assigning a variable to each of the values so that we can refer to them on the right side of the function definition.

To handle both "Birthday" and "Wedding" Anniversaries, we needed to provide *two* function definitions, one for each constructor. When `showAnniversary` is called, if the argument is a `Birthday` Anniversary, the first definition is used and the variables `name`, `month`, `date` and `year` are bound to its contents. If the argument is a `Wedding` Anniversary, then the second definition is used and the variables are bound in the same way. This process of using a different version of the function depending on the type of constructor is pretty much like what happens when we use a `case` statement or define a function piece-wise.

Note that the parentheses around the constructor name and the bound variables are mandatory; otherwise the compiler or interpreter would not take them as a single argument. Also, it is important to have it absolutely clear that the expression inside the parentheses is *not* a call to the constructor function, even though it may look just like one.

Exercises:
Note: The solution of this exercise is given near the end of the chapter, so we recommend that you attempt it before getting there.
Reread the function definitions above. Then look closer at the `showDate` helper function. We said it was provided "for the sake of code clarity", but there is a certain clumsiness in the way it is used. You have to pass three separate Int arguments to it, but these arguments are always linked to each other as part of a single date. It would make no sense to do things like passing the year, month and day values of the Anniversary in a different order, or to pass the month value twice and omit the day.

- Could we use what we've seen in this chapter so far to reduce this clumsiness?
- Declare a `Date` type which is composed of three Int, corresponding to year, month and day. Then, rewrite `showDate` so that it uses the new Date data type. What changes will then be needed in `showAnniversary` and the `Anniversary` for them to make use of Date?.

15.3 `type` for making type synonyms

As mentioned in the introduction of this module, code clarity is one of the motivations for using custom types. In that spirit, it could be nice to make it clear that the Strings in the Anniversary type are being used as *names* while still being able to manipulate them like ordinary Strings. This calls for a `type` declaration:

```
type Name = String
```

The code above says that a `Name` is now a synonym for a `String`. Any function that takes a `String` will now take a `Name` as well (and vice-versa: functions that take `Name` will accept any `String`). The right hand side of a `type` declaration can be a more complex type as well. For example, `String` itself is defined in the standard libraries as

```
type String = [Char]
```

We can do something similar for the list of anniversaries we made use of:

```
type AnniversaryBook = [Anniversary]
```

Type synonyms are mostly just a convenience. They help make the roles of types clearer or provide an alias to such things as complicated list or tuple types. It is largely a matter of personal discretion to decide how type synonyms should be deployed. Abuse of synonyms could make code confusing (for instance, picture a long program using multiple names for common types like Int or String simultaneously).

Incorporating the suggested type synonyms and the `Date` type we proposed in the exercise(*) of the previous section the code we've written so far looks like this:

((*) **last chance to try that exercise without looking at the spoilers.**)

```
type Name = String

data Anniversary =
   Birthday Name Date
   | Wedding Name Name Date

data Date = Date Int Int Int   -- Year, Month, Day

johnSmith :: Anniversary
johnSmith = Birthday "John Smith" (Date 1968 7 3)

smithWedding :: Anniversary
smithWedding = Wedding "John Smith" "Jane Smith" (Date 1987 3 4)

type AnniversaryBook = [Anniversary]

anniversariesOfJohnSmith :: AnniversaryBook
anniversariesOfJohnSmith = [johnSmith, smithWedding]

showDate :: Date -> String
showDate (Date y m d) = show y ++ "-" ++ show m ++ "-" ++ show d

showAnniversary :: Anniversary -> String
showAnniversary (Birthday name date) =
   name ++ " born " ++ showDate date
showAnniversary (Wedding name1 name2 date) =
   name1 ++ " married " ++ name2 ++ " on " ++ showDate date
```

Even in a simple example like this one, there is a noticeable gain in simplicity and clarity compared to the same task using only Ints, Strings, and corresponding lists.

Note that the `Date` type has a constructor function which is called `Date` as well. That is perfectly valid and indeed giving the constructor the same name of the type when there is just one constructor is good practice, as a simple way of making the role of the function obvious.

Note:
After these initial examples, the mechanics of using constructor functions may look a bit unwieldy, particularly if you're familiar with analogous features in other languages. There are syntactical constructs that make dealing with constructors more convenient. These will be dealt with later on, when we return to the topic of constructors and data types to explore them in detail.

16 Pattern matching

In the previous modules, we introduced and made occasional reference to pattern matching. Now that we have developed some familiarity with the language, it is time to take a proper, deeper look. We will kick-start the discussion with a condensed description, which we will expand upon throughout the chapter:

In pattern matching, we attempt to ***match*** *values against* ***patterns*** *and, if so desired,* ***bind*** *variables to successful matches.*

16.1 Analysing pattern matching

Pattern matching is virtually everywhere. For example, consider this definition of `map`:

```
map _ []     = []
map f (x:xs) = f x : map f xs
```

At surface level, there are four different patterns involved, two per equation.

- `f` is a pattern which matches *anything at all*, and binds the `f` variable to whatever is matched.
- `(x:xs)` is a pattern that matches a *non-empty list* which is formed by something (which gets bound to the `x` variable) which was cons'd (by the `(:)` function) onto something else (which gets bound to `xs`).
- `[]` is a pattern that matches *the empty list.* It doesn't bind any variables.
- `_` is the pattern which matches anything without binding (wildcard, ”don't care” pattern).

In the `(x:xs)` pattern, `x` and `xs` can be seen as sub-patterns used to match the parts of the list. Just like `f`, they match anything - though it is evident that if there is a successful match and `x` has type `a`, `xs` will have type `[a]`. Finally, these considerations imply that `xs` will also match an empty list, and so a one-element list matches `(x:xs)`.

From the above dissection, we can say pattern matching gives us a way to:

- *recognize values.* For instance, when `map` is called and the second argument matches `[]` the first equation for `map` is used instead of the second one.
- *bind variables* to the recognized values. In this case, the variables `f`, `x`, and `xs` are assigned to the values passed as arguments to `map` when the second equation is used, and so we can use these values through the variables in the right-hand side of `=`. As `_` and `[]` show, binding is not an essential part of pattern matching, but just a side effect of using variable names as patterns.
- *break down values into parts*, as the `(x:xs)` pattern does by binding two variables to parts (head and tail) of a matched argument (the non-empty list).

16.2 The connection with constructors

Despite the detailed analysis above, it may seem a little too magical how we break down a list as if we were undoing the effects of the (:) operator. Be careful: this process will not work with any arbitrary operator. For example, one might think of defining a function which uses (++) to chop off the first three elements of a list:

```
dropThree ([x,y,z] ++ xs) = xs
```

But that *will not work*. The function (++) is not allowed in patterns. In fact, most other functions that act on lists are similarly prohibited from pattern matching. Which functions, then, *are* allowed?

In one word, *constructors* – the functions used to build values of algebraic data types. Let us consider a random example:

```
data Foo = Bar | Baz Int
```

Here Bar and Baz are constructors for the type Foo. You *can* use them for pattern matching Foo values and bind variables to the Int value contained in a Foo constructed with Baz:

```
f :: Foo -> Int
f Bar     = 1
f (Baz x) = x - 1
```

This is exactly like showAnniversary and showDate in the Type declarations module. For instance:

```
data Date = Date Int Int Int   -- Year, Month, Day
showDate :: Date -> String
showDate (Date y m d) = show y ++ "-" ++ show m ++ "-" ++ show d
```

The (Date y m d) pattern in the left-hand side of the showDate definition matches a Date (built with the Date constructor) and binds the variables y, m and d to the contents of the Date value.

16.2.1 Why does it work with lists?

As for lists, they are no different from data-defined algebraic data types as far as pattern matching is concerned. It works as if lists were defined with this data declaration (note that the following isn't actually valid syntax: lists are actually too deeply ingrained into Haskell to be defined like this):

```
data [a] = [] | a : [a]
```

So the empty list, [] and the (:) function are constructors of the list datatype, and so you can pattern match with them. [] takes no arguments, and therefore no variables can be bound when it is used for pattern matching. (:) takes two arguments, the list head and tail, which may then have variables bound to them when the pattern is recognized.

```
Prelude> :t []
[] :: [a]
```

```
Prelude> :t (:)
(:) :: a -> [a] -> [a]
```

Furthermore, since `[x, y, z]` is just syntactic sugar for `x:y:z:[]`, we can achieve something like `dropThree` using pattern matching alone:

```
dropThree :: [a] -> [a]
dropThree (_:_:_:xs) = xs
dropThree _          = []
```

The first pattern will match any list with at least three elements. The catch-all second definition provides a reasonable default[1] when lists fail to match the main pattern, and thus prevents runtime crashes due to pattern match failure.

Note:
From the fact that we *could* write a `dropThree` function with bare pattern matching it doesn't follow that we *should* do so! Even though the solution is simple, it is still a waste of effort to code something this specific when we could just use Prelude and settle it with `drop 3 xs` instead. Mirroring what was said before about baking bare recursive functions, we might say: *don't get too excited about pattern matching either...*

16.2.2 Tuple constructors

Analogous considerations are valid for tuples. Our access to their components via pattern matching...

```
fstPlusSnd :: (Num a) => (a, a) -> a
fstPlusSnd (x, y) = x + y

norm3D :: (Floating a) => (a, a, a) -> a
norm3D (x, y, z) = sqrt (x^2 + y^2 + z^2)
```

... is granted by the existence of tuple constructors. For pairs, the constructor is the comma operator, `(,)`; for larger tuples there are `(,,)`; `(,,,)` and so on. These operators are slightly unusual in that we can't use them infix in the regular way; so `5 , 3` is not a valid way to write `(5, 3)`. All of them, however, can be used prefix, which is occasionally useful.

```
Prelude> (,) 5 3
(5,3)
Prelude> (,,,) "George" "John" "Paul" "Ringo"
("George","John","Paul","Ringo")
```

[1] Reasonable *for this particular task*, and only because it makes sense to expect that `dropThree` will give `[]` when applied to a list of, say, two elements. With a different problem, it might not be reasonable to return *any* list if the first match failed. In a later chapter, we will consider one simple way of dealing with such cases.

16.3 Matching literal values

As discussed earlier in the book, a simple piece-wise function definition like this one

```
f :: Int -> Int
f 0 = 1
f 1 = 5
f 2 = 2
f _ = -1
```

is performing pattern matching as well, matching the argument of `f` with the `Int` literals 0, 1 and 2, and finally with `_` . In general, numeric and character literals can be used in pattern matching on their own[2] as well as together with constructor patterns. For instance, this function

```
g :: [Int] -> Bool
g (0:[]) = False
g (0:xs) = True
g _ = False
```

will evaluate to False for the [0] list, to True if the list has 0 as first element and a non-empty tail and to False in all other cases. Also, lists with literal elements like [1,2,3], or even ”abc” (which is equivalent to [’a’,’b’,’c’]) can be used for pattern matching as well, since these forms are only syntactic sugar for the (:) constructor.

The above considerations are only valid for literal values, so the following will **not** work:

```
k = 1
--again, this won't work as expected
h :: Int -> Bool
h k = True
h _ = False
```

Exercises:

1. Test the flawed `h` function above in GHCi, with arguments equal to and different from 1. Then, explain what goes wrong.
2. In this section about pattern matching with literal values, we made no mention of the boolean values `True` and `False`, but we can do pattern matching with them as well, as demonstrated in the Next steps[a] chapter. Can you guess why we omitted them? (Hint: is there anything distinctive about the way we write boolean values?)

[a] Chapter 9.2 on page 50

[2] As perhaps could be expected, this kind of matching with literals is not constructor-based. Rather, there is an equality comparison behind the scenes

16.4 Syntax tricks

16.4.1 As-patterns

Sometimes, when matching a pattern with a value, it may be useful to bind a name to the whole value being matched. As-patterns allow exactly this: they are of the form *var@pattern* and have the additional effect to bind the name `var` to the whole value being matched by `pattern`. For instance, here is a toy variation on the map theme:

```
contrivedMap :: ([a] -> a -> b) -> [a] -> [b]
contrivedMap f [] = []
contrivedMap f list@(x:xs) = f list x : contrivedMap f xs
```

`contrivedMap` passes to the parameter function `f` not only `x` but also the undivided list used as argument of each recursive call. Writing it without as-patterns would have been a bit clunky because we would have to either use `head` or needlessly *reconstruct* the original value of `list`, i.e. actually evaluate `x:xs` on the right side:

```
contrivedMap :: ([a] -> a -> b) -> [a] -> [b]
contrivedMap f [] = []
contrivedMap f (x:xs) = f (x:xs) x : contrivedMap f xs
```

Exercises:
Implement `scanr`, as in the exercise in Lists III[a], but this time using an as-pattern.

[a] Chapter 14.2 on page 89

16.4.2 Introduction to records

For constructors with many elements, *records* provide a way of naming values in a datatype using the following syntax:

```
data Foo2 = Bar2 | Baz2 {bazNumber::Int, bazName::String}
```

Using records allows doing matching and binding only for the variables relevant to the function we're writing, making code much clearer:

```
h :: Foo2 -> Int
h Baz2 {bazName=name} = length name
h Bar2 {} = 0

x = Baz2 1 "Haskell"     -- construct by declaration order, try ":t Baz2" in
 GHCi
y = Baz2 {bazName = "Curry", bazNumber = 2} -- construct by name

h x -- 7
h y -- 5
```

Also, the `{}` pattern can be used for matching a constructor regardless of the datatype elements even if you don't use records in the `data` declaration:

```
data Foo = Bar | Baz Int
g :: Foo -> Bool
```

```
g Bar {} = True
g Baz {} = False
```

The function g does not have to be changed if we modify the number or the type of elements of the constructors Bar or Baz.

There are further advantages to using record syntax which we will cover in more details in the Named fields[3] section of the More on datatypes chapter.

16.5 Where we can use pattern matching

The short answer is that *wherever you can bind variables, you can pattern match.* Let us have a glance at such places we have seen before; a few more will be introduced in the following chapters.

16.5.1 Equations

The most obvious use case is the left-hand side of function definition equations, which were the subject of our examples so far.

```
map _ []     = []
map f (x:xs) = f x : map f xs
```

In the map definition we're doing pattern matching on the left hand side of both equations, and also binding variables on the second one.

16.5.2 let expressions and where clauses

Both let and where are ways of doing local variable bindings. As such, you can also use pattern matching in them. A simple example:

```
y =
  let
    (x:_) = map (*2) [1,2,3]
  in x + 5
```

Or, equivalently,

```
y = x + 5
  where
  (x:_) = map (*2) [1,2,3]
```

Here, x will be bound to the first element of map ((*) 2) [1,2,3]. y, therefore, will evaluate to $2+5=7$.

[3] Chapter 24.2 on page 139

16.5.3 List comprehensions

After the `|` in list comprehensions you can pattern match. This is actually extremely useful, and adds a lot to the expressiveness of comprehensions. Let's see how that works with a slightly more sophisticated example. Prelude provides a `Maybe` type which has the following constructors:

```
data Maybe a = Nothing | Just a
```

It is typically used to hold values resulting from an operation which may or may not succeed; if the operation succeeds, the `Just` constructor is used and the value is passed to it; otherwise `Nothing` is used.[4] The utility function `catMaybes` (which is available from Data.Maybe library module) takes a list of Maybes (which may contain both "Just" and "Nothing" Maybes), and retrieves the contained values by filtering out the `Nothing` values and getting rid of the `Just` wrappers of the `Just x`. Writing it with list comprehensions is very straightforward:

```
catMaybes :: [Maybe a] -> [a]
catMaybes ms = [ x | Just x <- ms ]
```

Another nice thing about using a list comprehension for this task is that if the pattern match fails (that is, it meets a Nothing) it just moves on to the next element in `ms`, thus avoiding the need of explicitly handling constructors we are not interested in with alternate function definitions.[5]

16.5.4 do blocks

Within a `do` block like the ones we used in the Simple input and output[6] chapter, we can pattern match with the left-hand side of the left arrow variable bindings:

```
putFirstChar = do
    (c:_) <- getLine
    putStrLn [c]
```

Furthermore, the `let` bindings in `do` blocks are, as far as pattern matching is concerned, just the same as the "real" let expressions.

[4] The canonical example of such an operation is looking up values in a *dictionary* - which might just be a `[(a, b)]` list with the tuples being key-value pairs, or a more sophisticated implementation. In any case, if we, given an arbitrary key, try to retrieve a value there is no guarantee we will actually find a value associated to the key.

[5] The reason why it works this way instead of crashing out on a pattern matching failure has to do with the real nature of list comprehensions: They are actually wrappers for the list monad. We will eventually explain what that means when we discuss monads.

[6] Chapter 10 on page 57

17 Control structures

Haskell offers several ways of expressing a choice between different values. We explored some of them in the Haskell Basics chapters. This section will bring together what we have seen thus far, discuss some finer points, and introduce a new control structure.

17.1 `if` and guards revisited

We have already met these constructs. The syntax for `if` expressions is:

```
if <condition> then <true-value> else <false-value>
```

`<condition>` is an expression which evaluates to a boolean. If the `<condition>` is `True` then the `<true-value>` is returned, otherwise the `<false-value>` is returned. Note that in Haskell `if` is an expression (which is converted to a value) and not a statement (which is executed) as in many imperative languages.[1] As a consequence, the `else` is *mandatory* in Haskell. Since `if` is an expression, it must evaluate to a result whether the condition is true or false, and the `else` ensures this. Furthermore, `<true-value>` and `<false-value>` must evaluate to the same type, which will be the type of the whole if expression.

When `if` expressions are split across multiple lines, they are usually indented by aligning `else`s with `then`s, rather than with `if`s. A common style looks like this:

```
describeLetter :: Char -> String
describeLetter c =
    if c >= 'a' && c <= 'z'
        then "Lower case"
        else if c >= 'A' && c <= 'Z'
            then "Upper case"
            else "Not an ASCII letter"
```

Guards and top-level `if` expressions are mostly interchangeable. With guards, the example above is a little neater:

```
describeLetter :: Char -> String
describeLetter c
   | c >= 'a' && c <= 'z' = "Lower case"
   | c >= 'A' && c <= 'Z' = "Upper case"
   | otherwise            = "Not an ASCII letter"
```

Remember that `otherwise` is just an alias to `True`, and thus the last guard is a catch-all, playing the role of the final `else` of the `if` expression.

[1] If you have programmed in C or Java, you will recognize Haskell's if/then/else as an equivalent to the ternary conditional operator `?:` .

Guards are evaluated in the order they appear. Consider a set up like the following:

```
f (pattern1) | predicate1 = w
             | predicate2 = x

f (pattern2) | predicate3 = y
             | predicate4 = z
```

Here, the argument of `f` will be pattern-matched against pattern1. If it succeeds, then we proceed to the first set of guards: if predicate1 evaluates to `True`, then `w` is returned. If not, then predicate2 is evaluated; and if it is true `x` is returned. Again, if not, then we proceed to the next case and try to match the argument against pattern2, repeating the guards procedure with predicate3 and predicate4. (Of course, if neither pattern matches or neither predicate is true for the matching pattern there will be a runtime error. Regardless of the chosen control structure, it is important to ensure all cases are covered.)

17.1.1 Embedding `if` expressions

A handy consequence of `if` constructs being *expressions* is that they can be placed anywhere a Haskell expression could be, allowing us to write code like this:

```
g x y = (if x == 0 then 1 else sin x / x) * y
```

Note that we wrote the `if` expression without line breaks for maximum terseness. Unlike `if` expressions, guard blocks are not expressions; and so a `let` or a `where` definition is the closest we can get to this style when using them. Needless to say, more complicated one-line `if` expressions would be hard to read, making `let` and `where` attractive options in such cases.

17.2 `case` expressions

One control structure we haven't talked about yet are `case` expressions. They are to piece-wise function definitions what `if` expressions are to guards. Take this simple piece-wise definition:

```
f 0 = 18
f 1 = 15
f 2 = 12
f x = 12 - x
```

It is equivalent to - and, indeed, syntactic sugar for:

```
f x =
    case x of
        0 -> 18
        1 -> 15
        2 -> 12
        _ -> 12 - x
```

Whatever definition we pick, the same happens when `f` is called: The argument `x` is matched against all of the patterns in order; and on the first match the expression on the right-hand side of the corresponding equal sign (in the piece-wise version) or arrow (in the `case` version) is evaluated. Note that in this case expression there is no need to write x in the pattern; the wildcard pattern _ gives the same effect.[2]

Indentation is important when using case. The cases must be indented further to the right than the beginning of the line containing the of keyword, and all cases must have the same indentation. For the sake of illustration, here are two other valid layouts for a case expression:

```
f x = case x of
    0 -> 18
    1 -> 15
    2 -> 12
    _ -> 12 - x
```

```
f x = case x of 0 -> 18
                1 -> 15
                2 -> 12
                _ -> 12 - x
```

Since the left hand side of any case branch is just a pattern, it can also be used for binding, exactly like in piece-wise function definitions:[3]

```
describeString :: String -> String
describeString str =
  case str of
    (x:xs) -> "The first character of the string is: " ++ [x] ++ "; and " ++
              "there are " ++ show (length xs) ++ " more characters in it."
    []     -> "This is an empty string."
```

This function describes some properties of str using a human-readable string. Using case syntax to bind variables to the head and tail of our list is convenient here, but you could also do this with an if-statement (with a condition of null str to pick the empty string case).

Finally, just like if expressions (and unlike piece-wise definitions), case expressions can be embedded anywhere another expression would fit:

```
data Colour = Black | White | RGB Int Int Int

describeBlackOrWhite :: Colour -> String
describeBlackOrWhite c =
  "This colour is"
  ++ case c of
       Black           -> " black"
       White           -> " white"
       RGB 0 0 0       -> " black"
       RGB 255 255 255 -> " white"
       _               -> "... uh... something else"
  ++ ", yeah?"
```

The case block above fits in as any string would. Writing describeBlackOrWhite this way makes let/where unnecessary (although the resulting definition is not as readable).

[2] To see why this is so, consider our discussion of matching and binding in the ../Pattern matching/ ^{Chapter16 on page 99} section

[3] Thus, case statements are a lot more versatile than most of the superficially similar switch/case statements in imperative languages which are typically restricted to equality tests on integral primitive types.

Exercises:
Use a `case` statement to implement a `fakeIf` function which could be used as a replacement to the familiar `if` expressions.

17.3 Controlling actions, revisited

In the final part of this chapter, we will introduce a few extra points about control structures while revisiting the discussions in the ”Simple input and output” chapter. There, in the Controlling actions[4] section, we used the following function to show how to execute actions conditionally within a `do` block using `if` expressions:

```
doGuessing num = do
   putStrLn "Enter your guess:"
   guess <- getLine
   if (read guess) < num
     then do putStrLn "Too low!"
             doGuessing num
     else if (read guess) > num
            then do putStrLn "Too high!"
                    doGuessing num
            else do putStrLn "You Win!"
```

We can write the same `doGuessing` function using a `case`statement. To do this, we first introduce the Prelude function `compare` which takes two values of the same type (in the `Ord`class) and returns a value of type `Ordering` — namely one of `GT`, `LT`, `EQ`, depending on whether the first is greater than, less than, or equal to the second.

```
doGuessing num = do
  putStrLn "Enter your guess:"
  guess <- getLine
  case compare (read guess) num of
    LT -> do putStrLn "Too low!"
             doGuessing num
    GT -> do putStrLn "Too high!"
             doGuessing num
    EQ -> putStrLn "You Win!"
```

The `do`s after the `->`s are necessary on the first two options, because we are sequencing actions within each case.

17.3.1 A note about `return`

Now, we are going to dispel a possible source of confusion. In a typical imperative language (C, for example) an implementation of `doGuessing` might look like the following (if you don't know C, don't worry with the details, just follow the if-else chain):

```
void doGuessing(int num) {
  printf("Enter your guess:");
  int guess = atoi(readLine());
  if (guess == num) {
```

[4] Chapter 10.2.2 on page 60

```
    printf("You win!\n");
    return ();
  }

  // we won't get here if guess == num
  if (guess < num) {
    printf("Too low!\n");
    doGuessing(num);
  } else {
    printf("Too high!\n");
    doGuessing(num);
  }
}
```

This `doGuessing` first tests the equality case, which does not lead to a new call of `doGuessing`, and the `if` has no accompanying `else`. If the guess was right, a `return` statement is used to exit the function at once, skipping the other cases. Now, going back to Haskell, action sequencing in `do` blocks looks a lot like imperative code, and furthermore there actually *is* a `return` in Prelude. Then, knowing that `case`statements (unlike `if` statements) do not force us to cover all cases, one might be tempted to write a literal translation of the C code above (try running it if you are curious)...

```
doGuessing num = do
  putStrLn "Enter your guess:"
  guess <- getLine
  case compare (read guess) num of
    EQ -> do putStrLn "You win!"
             return ()

  -- we don't expect to get here if guess == num
  if (read guess < num)
    then do putStrLn "Too low!";
            doGuessing num
    else do putStrLn "Too high!";
            doGuessing num
```

... but it won't work! If you guess correctly, the function will first print ”You win!,” but it *will not exit* at the `return ()`. Instead, the program will continue to the `if` expression and check whether `guess` is less than `num`. Of course it is not, so the else branch is taken, and it will print ”Too high!” and then ask you to guess again. Things aren't any better with an incorrect guess: it will try to evaluate the case statement and get either `LT` or `GT` as the result of the `compare`. In either case, it won't have a pattern that matches, and the program will fail immediately with an exception (as usual, the incomplete `case` alone should be enough to raise suspicion).

The problem here is that `return` is not at all equivalent to the C (or Java etc.) statement with the same name. For our immediate purposes, we can say that `return` is a *function.*[5] The `return ()` in particular evaluates to an action which does nothing. `return` *does not affect the control flow at all.* In the correct guess case, the case statement evaluates to `return ()`, an action of type `IO ()`, and execution just follows along normally.

[5] *Superfluous note*: somewhat closer to a proper explanation, we might say `return` is a function which takes a value and makes it into an action which, when evaluated, gives the original value. A `return "strawberry"` within one of the `do` blocks we are dealing with would have type `IO String` - the same type as `getLine`. Do not worry if that doesn't make sense for now; you will understand what `return` really does when we *actually* start discussing monads further ahead on the book.

The bottom line is that while actions and do blocks resemble imperative code, they must be dealt with on their own terms - Haskell terms.

Exercises:

1. Redo the ”Haskell greeting” exercise in Simple input and output/Controlling actions[a], this time using a case statement.
2. What does the following program print out? And why?

```
main =
 do x <- getX
    putStrLn x

getX =
 do return "My Shangri-La"
    return "beneath"
    return "the summer moon"
    return "I will"
    return "return"
    return "again"
```

[a] Chapter 10.2.2 on page 60

18 More on functions

Here are several nice features that make using functions easier.

18.1 let and where revisited

As discussed in earlier chapters, `let` and `where` are useful in local function definitions. Here, `sumStr` calls `addStr` function:

```
addStr :: Float -> String -> Float
addStr x str = x + read str

sumStr :: [String] -> Float
sumStr = foldl addStr 0.0
```

But what if we never need `addStr` anywhere else? Then we could rewrite `sumStr` using local bindings. We can do that either with a `let` binding...

```
sumStr =
   let addStr x str = x + read str
   in foldl addStr 0.0
```

... or with a `where` clause...

```
sumStr = foldl addStr 0.0
   where addStr x str = x + read str
```

... and the difference appears to be just a question of style: Do we prefer the bindings to come before or after the rest of the definition?

However, there is another important difference between `let` and `where`. The `let...in` construct is an expression just like if/then/else. In contrast, `where` clauses are like guards and so are not expressions. Thus, `let` bindings can be used within complex expressions:

```
f x =
    if x > 0
        then (let lsq = (log x) ^ 2 in tan lsq) * sin x
        else 0
```

The expression within the outer parentheses is self-contained, and evaluates to the tangent of the square of the logarithm of `x`. Note that the scope of `lsq` does not extend beyond the parentheses; so changing the then-branch to

```
        then (let lsq = (log x) ^ 2 in tan lsq) * (sin x + lsq)
```

does not work without dropping the parentheses around the let.

Despite not being full expressions, where clauses *can* be incorporated into case expressions:

```
describeColour c =
   "This colour "
   ++ case c of
          Black -> "is black"
          White -> "is white"
          RGB red green blue -> " has an average of the components of " ++ show
 av
             where av = (red + green + blue) `div` 3
   ++ ", yeah?"
```

In this example, the indentation of the where clause sets the scope of the av variable so that it only exists as far as the RGB red green blue case is concerned. Placing it at the same indentation of the cases would make it available for all cases. Here is an example with guards:

```
doStuff :: Int -> String
doStuff x
  | x < 3     = report "less than three"
  | otherwise = report "normal"
  where
    report y = "the input is " ++ y
```

Note that since there is one equals sign for each guard there is no place we could put a let expression which would be in scope of all guards in the manner of the where clause. So this is a situation in which where is particularly convenient.

18.2 Anonymous Functions - lambdas

Why create a formal name for a function like addStr when it only exists within another function's definition, never to be used again? Instead, we can make it an anonymous function also known as a "lambda function". Then, sumStr could be defined like this:

```
sumStr = foldl (\ x str -> x + read str) 0.0
```

The expression in the parentheses is a lambda function. The backslash is used as the nearest ASCII equivalent to the Greek letter lambda (λ). This lambda function takes two arguments, x and str, and it evaluates to "x + read str". So, the sumStr presented just above is precisely the same as the one that used addStr in a let binding.

Lambdas are handy for writing one-off functions to be used with maps, folds and their siblings, especially where the function in question is simple (beware of cramming complicated expressions in a lambda — it can hurt readability).

Since variables are being bound in a lambda expression (to the arguments, just like in a regular function definition), pattern matching can be used in them as well. A trivial example would be redefining tail with a lambda:

```
tail' = (\ (_:xs) -> xs)
```

Note: Since lambdas are a special character in Haskell, the \ on its own will be treated as the function and whatever non-space character is next will be the variable for the first argument. It is still good form to put a space between the lambda and the argument as in normal function syntax (especially to make things clearer when a lambda takes more than one argument).

18.3 Operators

In Haskell, any function that takes two arguments and has a name consisting entirely of non-alphanumeric characters is considered an *operator*. The most common examples are the arithmetical ones like addition (+) and subtraction (-). Unlike other functions, operators are normally used infix (written between the two arguments). All operators can also be surrounded with parentheses and then used prefix like other functions:

```
-- these are the same:
2 + 4
(+) 2 4
```

We can define new operators in the usual way as other functions — just don't use any alphanumeric characters in their names. For example, here's the set-difference definition from `Data.List`:

```
(\\) :: (Eq a) => [a] -> [a] -> [a]
xs \\ ys = foldl (\zs y -> delete y zs) xs ys
```

As the example above shows, operators can be defined infix as well. The same definition written as prefix also works:

```
(\\) xs ys = foldl (\zs y -> delete y zs) xs ys
```

Note that the type declarations for operators have no infix version and must be written with the parentheses.

18.3.1 Sections

Sections are a nifty piece of syntactical sugar that can be used with operators. An operator within parentheses and flanked by one of its arguments...

```
(2+) 4
(+4) 2
```

... is a new function in its own right. `(2+)`, for instance, has the type `(Num a) => a -> a`. We can pass sections to other functions, e.g. `map (+2) [1..4] == [3..6]`. For another example, we can add an extra flourish to the `multiplyList` function we wrote back in More about lists:

```
multiplyList :: Integer -> [Integer] -> [Integer]
multiplyList m = map (m*)
```

If you have a ”normal” prefix function and want to use it as an operator, simply surround it with backticks:

```
1 `elem` [1..4]
```

This is called making the function *infix*. It’s normally done for readability purposes: ``1 `elem` [1..4]`` reads better than `elem 1 [1..4]`. You can also define functions infix:

```
elem :: (Eq a) => a -> [a] -> Bool
x `elem` xs = any (==x) xs
```

But once again notice that the type signature stays with the prefix style.

Sections even work with infix functions:

```
(1 `elem`) [1..4]
(`elem` [1..4]) 1
```

Of course, remember that you can only make binary functions (that is, those that take two arguments) infix.

Exercises:

- Lambdas are a nice way to avoid defining unnecessary separate functions. Convert the following let- or where-bindings to lambdas:
  - `map f xs where f x = x * 2 + 3`
  - `let f x y = read x + y in foldr f 1 xs`
- Sections are just syntactic sugar for lambda operations. I.e. `(+2)` is equivalent to `\x -> x + 2`. What would the following sections ’desugar’ to? What would be their types?
  - `(4+)`
  - ``(1 `elem`)``
  - ``(`notElem` "abc")``

19 Higher-order functions

At the heart of functional programming is the idea that functions are *just like any other value*. The power of functional style comes from handling functions themselves as regular values, i.e. by passing functions to other functions and returning them from functions. A function that takes another function (or several functions) as an argument is called a *higher-order function*. They can be found pretty much anywhere in a Haskell program; and indeed we have already met some of them, such as `map` and the various folds. We saw commonplace examples of higher-order functions when discussing `map` in Lists II[1]. Now, we are going to explore some common ways of writing code that manipulates functions.

19.1 A sorting algorithm

For a concrete example, we will consider the task of sorting a list. *Quicksort* is a well-known recursive sorting algorithm. To apply its sorting strategy to a list, we first choose one element and then divide the rest of the list into (A) those elements that should go before the chosen element, (B) those elements equal to the chosen one, and (C) those that should go after. Then, we apply the same algorithm to the unsorted (A) and (C) lists. After enough recursive sorting, we concatenate everything back together and have a final sorted list. That strategy can be translated into a Haskell implementation in a very simple way.

```haskell
-- Type signature: any list with elements in the Ord class can be sorted.
quickSort :: (Ord a) => [a] -> [a]
-- Base case:
-- If the list is empty, there is nothing to do.
quickSort [] = []

-- The recursive case:
-- We pick the first element as our "pivot", the rest is to be sorted.
-- Note how the pivot itself ends up included in the middle part.
quickSort (x : xs) = (quickSort less) ++ (x : equal) ++ (quickSort more)
    where
        less = filter (< x) xs
        equal = filter (== x) xs
        more = filter (> x) xs
```

It should be pointed out that our `quickSort` is rather naïve. A more efficient implementation would avoid the three passes through `filter` at each recursive step and not use `(++)` to build the sorted list. Furthermore, unlike our implementation, the *original* quicksort algorithm does the sorting in-place using mutability.[2] We will ignore such concerns for now,

[1] Chapter 13 on page 77

[2] The ”true”, in-place quicksort *can* be done in Haskell, but it requires some rather advanced tools that we will not discuss in the Beginners' Track.

as we are more interested in the usage patterns of sorting functions, rather than in exact implementation.

19.1.1 The `Ord` class

Almost all the basic data types in Haskell are members of the `Ord` class, which is for ordering tests what `Eq` is for equality tests. The `Ord` class defines which ordering is the ”natural” one for a given type. It provides a function called `compare`, with type:

```
compare :: (Ord a) => a -> a -> Ordering
```

`compare` takes two values and compares them, returning an `Ordering` value, which is `LT` if the first value is *less than* the second, `EQ` if it is *equal* and `GT` if it is *greater than.* For an `Ord` type, `(<)`, `(==)` from `Eq` and `(>)` can be seen as shortcuts to `compare` that check for one of the three possibilities and return a `Bool` to indicate whether the specified ordering is true according to the `Ord` specification for that type. Note that each of the tests we use with `filter` in the definition of `quickSort` corresponds to one of the possible results of `compare`, and so we might have written, for instance, `less` as ``less = filter (\y -> y `compare` x == LT) xs``.

19.2 Choosing how to compare

With `quickSort`, sorting any list with elements in the `Ord` class is easy. Suppose we have a list of `String` and we want to sort them; we just apply `quickSort` to the list. For the rest of this chapter, we will use a pseudo-dictionary of just a few words (but dictionaries with thousands of words would work just as well):

```
dictionary = ["I", "have", "a", "thing", "for", "Linux"]
```

`quickSort dictionary` returns:

```
["I", "Linux", "a", "for", "have", "thing"]
```

As you can see, capitalization is considered for sorting by default. Haskell `String`s are lists of Unicode characters. Unicode (and almost all other encodings of characters) specifies that the character code for capital letters are less than the lower case letters. So ”Z” is less than ”a”.

To get a proper dictionary-like sorting, we need a case insensitive `quickSort`. To achieve that, we can take a hint from the discussion of `compare` just above. The recursive case of `quickSort` can be rewritten as:

```
quickSort compare (x : xs) = (quickSort compare less) ++ (x : equal) ++
 (quickSort compare more)
    where
        less  = filter (\y -> y `compare` x == LT) xs
        equal = filter (\y -> y `compare` x == EQ) xs
        more  = filter (\y -> y `compare` x == GT) xs
```

While this version is less tidy than the original one, it makes it obvious that the ordering of the elements hinges entirely on the `compare` function. That means we only need to replace `compare` with an `(Ord a) => a -> a -> Ordering` function of our choice. Therefore, our updated `quickSort'` is a higher-order function which takes a comparison function along with the list to sort.

```
quickSort' :: (Ord a) => (a -> a -> Ordering) -> [a] -> [a]
-- No matter how we compare two things the base case doesn't change,
-- so we use the _ "wildcard" to ignore the comparison function.
quickSort' _ [] = []

-- c is our comparison function
quickSort' c (x : xs) = (quickSort' c less) ++ (x : equal) ++ (quickSort' c
 more)
    where
        less  = filter (\y -> y `c` x == LT) xs
        equal = filter (\y -> y `c` x == EQ) xs
        more  = filter (\y -> y `c` x == GT) xs
```

We can reuse our `quickSort'` function to serve many different purposes.

If we wanted a *descending* order, we could just reverse our original sorted list with `reverse (quickSort dictionary)`. Yet to actually do the initial sort descending, we could supply `quickSort'` with a comparison function that returns the opposite of the usual `Ordering`.

```
-- the usual ordering uses the compare function from the Ord class
usual = compare

-- the descending ordering, note we flip the order of the arguments to compare
descending x y = compare y x

-- the case-insensitive version is left as an exercise!
insensitive = ...
-- How can we do case-insensitive comparisons without making a big list of all
 possible cases?
```

Note:
`Data.List` offers a `sort` function for sorting lists. It does not use quicksort; rather, it uses an efficient implementation of an algorithm called *mergesort.* `Data.List` also includes `sortBy`, which takes a custom comparison function just like our `quickSort'`

Exercises:
Write `insensitive`, such that `quickSort' insensitive dictionary` gives `["a", "for", "have", "I", "Linux", "thing"]`.

19.3 Higher-Order Functions and Types

The concept of *currying* (the generating of intermediate functions on the way toward a final result) was first introduced in the earlier chapter ”More about lists”. This is a good place to revisit how currying works.

Our `quickSort'` has type `(a -> a -> Ordering) -> [a] -> [a]`.

Most of the time, the type of a higher-order function provides a guideline about how to use it. A straightforward way of reading the type signature would be ”quickSort' takes, as its first argument, a function that gives an ordering of two as. Its second argument is a list of as. Finally, it returns a new list of as”. This is enough to correctly guess that it uses the given ordering function to sort the list.

Note that the parentheses surrounding a -> a -> Ordering are mandatory. They specify that a -> a -> Ordering forms a single argument that happens to be a function.

Without the parentheses, we would get a -> a -> Ordering -> [a] -> [a] which accepts four arguments (none of which are themselves functions) instead of the desired two, and that wouldn't work as desired.

Remember that the -> operator is right-associative. Thus, our erroneous type signature a -> a -> Ordering -> [a] -> [a] means the same thing as a -> (a -> (Ordering -> ([a] -> [a]))).

Given that -> is right-associative, the explicitly grouped version of the correct quickSort' signature is actually (a -> a -> Ordering) -> ([a] -> [a]). This makes perfect sense. Our original quickSort lacking the adjustable comparison function argument was of type [a] -> [a]. It took a list and sorted it. Our new quickSort' is simply a function that generates quickSort style functions! If we plug in compare for the (a -> a -> Ordering) part, then we just return our original simple quickSort function. If we use a different comparison function for the argument, we generate a *different* variety of a quickSort function.

Of course, if we not only give a comparison function as an argument but also feed in an actual list to sort, then the final result is not the new quickSort-style function; instead, it continues on and passes the list to the new function and returns the sorted list as our final result.

Exercises:
(Challenging) The following exercise combines what you have learned about higher order functions, recursion and I/O. We are going to recreate what is known in imperative languages as a *for loop*. Implement a function

```
for :: a -> (a -> Bool) -> (a -> a) -> (a -> IO ()) -> IO ()
for i p f job = -- ???
```

An example of how this function would be used might be

```
for 1 (<10) (+1) print
```

which prints the numbers 1 to 9 on the screen.
The desired behaviour of `for` is: starting from an initial value `i`, `for` executes `job i`. It then uses `f` to modify this value and checks to see if the modified value `f i` satisfies some condition `p`. If it doesn't, it stops; otherwise, the for loop continues, using the modified `f i` in place of `i`.

1. Implement the for loop in Haskell.
2. The paragraph just above gives an imperative description of the for loop. Describe your implementation in more functional terms.
   Some more challenging exercises you could try
3. Consider a task like "print the list of numbers from 1 to 10". Given that `print` is a function, and we can apply it to a list of numbers, using `map` sounds like the natural thing to do. But would it actually work?
4. Implement a function `sequenceIO :: [IO a] -> IO [a]`. Given a list of actions, this function runs each of the actions in order and returns all their results as a list.
5. Implement a function `mapIO :: (a -> IO b) -> [a] -> IO [b]` which given a function of type `a -> IO b` and a list of type `[a]`, runs that action on each item in the list, and returns the results.
   This exercise was inspired from a blog post by osfameron. No peeking!

19.4 Function manipulation

We will close the chapter by discussing a few examples of common and useful general-purpose higher-order functions. Familiarity with these will greatly enhance your skill at both writing and reading Haskell code.

19.4.1 Flipping arguments

`flip` is a handy little Prelude function. It takes a function of two arguments and returns a version of the same function with the arguments swapped.

```
flip :: (a -> b -> c) -> b -> a -> c
```

`flip` in use:

```
Prelude> (flip (/)) 3 1
0.3333333333333333
Prelude> (flip map) [1,2,3] (*2)
[2,4,6]
```

We could have used flip to write a point-free version of the `descending` comparing function from the quickSort example:

```
descending = flip compare
```

`flip` is particularly useful when we want to pass a function with two arguments of different types to another function and the arguments are in the wrong order with respect to the signature of the higher-order function.

19.4.2 Composition

The `(.)` composition operator is another higher-order function. It has the signature:

```
(.) :: (b -> c) -> (a -> b) -> a -> c
```

`(.)` takes two functions as arguments and returns a new function which applies both the second argument and then the first.

Composition and higher-order functions provide a range of powerful tricks. For a tiny sample, first consider the `inits` function, defined in the module `Data.List`. Quoting the documentation, it ”returns all initial segments of the argument, shortest first”, so that:

```
Prelude Data.List> inits [1,2,3]
[[],[1],[1,2],[1,2,3]]
```

We can provide a one-line implementation for `inits` (written point-free for extra dramatic effect) using only the following higher-order functions from Prelude: `flip`, `scanl`, `(.)` and `map`:

```
myInits :: [a] -> [[a]]
myInits = map reverse . scanl (flip (:)) []
```

Swallowing a definition so condensed may look daunting at first, so analyze it slowly, bit by bit, recalling what each function does and using the type signatures as a guide.

The definition of `myInits` is super concise and clean with use of parentheses kept to a bare minimum. Naturally, if one goes overboard with composition by writing mile-long `(.)` chains, things will get confusing; but, when deployed reasonably, these point-free styles shine. Furthermore, the implementation is quite ”high level”: we do not deal explicitly with details like pattern matching or recursion; the functions we deployed — both the higher-order ones and their functional arguments — take care of such plumbing.

19.4.3 Application

`($)` is a curious higher-order operator. Its type is:

```
($) :: (a -> b) -> a -> b
```

It takes a function as its first argument, and *all* it does is to apply the function to the second argument, so that, for instance, `(head $ "abc") == (head "abc")`.

You might think that `($)` is completely useless! However, there are two interesting points about it. First, `($)` has very low precedence,[3] unlike regular function application which has the highest precedence. In effect, that means we can avoid confusing nesting of parentheses by breaking precedence with `$`. We write a non-point-free version of `myInits` without adding new parentheses:

```
myInits :: [a] -> [[a]]
myInits xs = map reverse . scanl (flip (:)) [] $ xs
```

Furthermore, as `($)` is just a function which happens to apply functions, and functions are just values, we can write intriguing expressions such as:

```
map ($ 2) [(2*), (4*), (8*)]
```

(Yes, that is a list of functions, and it is perfectly legal.)

19.4.4 `uncurry` and `curry`

As the name suggests, `uncurry` is a function that undoes currying; that is, it converts a function of two arguments into a function that takes a pair as its only argument.

```
uncurry :: (a -> b -> c) -> (a, b) -> c
```

```
Prelude> let addPair = uncurry (+)
Prelude> addPair (2, 3)
5
```

One interesting use of `uncurry` occasionally seen in the wild is in combination with `($)`, so that the first element of a pair is applied to the second.

```
Prelude> uncurry ($) (reverse, "stressed")
"desserts"
```

There is also `curry`, which is the opposite of `uncurry`.

```
curry :: ((a, b) -> c) -> a -> b -> c
```

```
Prelude> curry addPair 2 3 -- addPair as in the earlier example.
5
```

Because most Haskell functions are already curried, `curry` is nowhere near as common as `uncurry`.

[3] As a reminder, precedence here is meant in the same sense that `*` has higher precedence (i.e. is evaluated first) than `+` in mathematics.

19.4.5 `id` and `const`

Finally, we should mention two functions which, while not higher-order functions themselves, are most often used as arguments to higher-order functions. `id`, the *identity* function, is a function with type `a -> a` that returns its argument unchanged.

```
Prelude> id "Hello"
"Hello"
```

Similar in spirit to `id`, `const` is an `a -> b -> a` function that works like this:

```
Prelude> const "Hello" "world"
"Hello"
```

`const` takes two arguments, discards the second and returns the first. Seen as a function of one argument, `a -> (b -> a)`, it returns a constant function, which always returns the same value no matter what argument it is given.

`id` and `const` might appear worthless at first. However, when dealing with higher-order functions it is sometimes necessary to pass a dummy function, be it one that does nothing with its argument or one that always returns the same value. `id` and `const` give us convenient dummy functions for such cases.

Exercises:

1. Write implementations for `curry`, `uncurry` and `const`.
2. Describe what the following functions do without testing them:
   - `uncurry const`
   - `curry fst`
   - `curry swap`, where `swap :: (a, b) -> (b, a)` swaps the elements of a pair. (`swap` can be found in `Data.Tuple`.)
3. *(Very hard)* Use `foldr` to implement `foldl`. Hint: begin by reviewing the sections about `foldr` and `foldl` in Lists III[a]. There are two solutions; one is easier but relatively boring and the other is truly interesting. For the interesting one, think carefully about how you would go about composing all functions in a list.

[a] Chapter 14.1.1 on page 86

20 Using GHCi effectively

GHCi assists in several ways toward more efficient work. Here, we will discuss some of the best practices for using GHCi.

20.1 User interface

20.1.1 Tab completion

As in many other terminal programs, you can enter some starting text in GHCi and then hit the Tab key to be presented with a list of all possibilities that start with what you've written so far. When there is only one possibility, using Tab will auto-complete the string. For example `fol<Tab>` will append letter "d" (since nothing exists with "fol" other than items that start with "fold"). A second `Tab` will list the four functions included in Prelude: `foldl`, `foldl1`, `foldr`, and `foldr1`. More options may show if you have already imported additional modules.

Tab completion works also when you are loading a file with your program into GHCi. For example, after typing `:l fi<Tab>`, you will be presented with all files that start with "fi" that are present in the current directory (the one you were in when you launched GHCi).

The same also applies when you are importing modules, after typing `:m +Da<Tab>` or `import Da<Tab>`, you will be presented with all modules that start with "Da" present in installed packages.

20.1.2 ": commands"

On GHCi command line, commands for the interpreter start with the character ":" (colon).

- `:help` or `:h` -- prints a list of all available commands.
- `:load` or `:l` -- loads a given file into GHCi (you must include the filename with the command).
- `:reload` or `:r` -- reloads whatever file had been loaded most recently (useful after changes to the file).
- `:type` or `:t` -- prints the type of a given expression included with the command
- `:module` or `:m` -- loads a given module (include the module name with the command). You can also *un*load a module by adding a `-` symbol before the module name.
- `:browse` -- gives the type signatures for all functions available from a given module.

Here again, you can use `Tab` to see the list of commands, type `:Tab` to see all possible commands.

20.1.3 Timing Functions in GHCi

GHCi provides a basic way to measure how much time a function takes to run, which can be useful for to find out which version of a function runs fastest (such as when there are multiple ways to define something to get the same effective result).

1. Type `:set +s` into the ghci command line.
2. run the function(s) you are testing. The time the function took to run will be displayed after GHCi outputs the results of the function.

20.1.4 Multi-line Input

If you are trying to define a function that takes up multiple lines, or if you want to type a do block into ghci (without writing a file that you then import), there is an easy way to do this:

1. Begin a new line with `:{`
2. Type in your code. Press enter when you need a new line.
3. Type `:}` to end the multi-line input.

For example:

```
*Main> :{
*Main| let askname = do
*Main|               putStrLn "What is your name?"
*Main|               name <- getLine
*Main|               putStrLn $ "Hello " ++ name
*Main| :}
*Main>
```

The same can be accomplished by using `:set +m` command (allow multi-line commands). In this case, an empty line will end the block.

In addition, line breaks in ghci commands can be separated by `;`, like this:

```
   *Main> let askname1 = do ; putStrLn "what is your name?" ; name <- getLine ;
putStrLn $ "Hello " ++ name
```

21 Intermediate Haskell

22 Modules

Modules are the primary means of organizing Haskell code. We met them in passing when using `import` statements to put library functions into scope. Beyond allowing us to make better use of libraries, knowledge of modules will help us to shape our own programs and create standalone programs which can be executed independently of GHCi (incidentally, that is the topic of the very next chapter, ../Standalone programs/[1]).

22.1 Modules

Haskell modules[2] are a useful way to group a set of related functionalities into a single package and manage different functions that may have the same names. The module definition is the first thing that goes in your Haskell file.

A basic module definition looks like:

```
module YourModule where
```

Note that

1. the name of the module begins with a capital letter;
2. each file contains only one module.

The name of the file is the name of the module plus the `.hs` file extension. Any dots '`.`' in the module name are changed for directories.[3] So the module `YourModule` would be in the file `YourModule.hs` while a module `Foo.Bar` would be in the file `Foo/Bar.hs` or `Foo\Bar.hs`. Since the module name must begin with a capital letter, the file name must also start with a capital letter.

22.2 Importing

Modules can themselves import functions from other modules. That is, in between the module declaration and the rest of your code, you may include some import declarations such as

[1] `https://en.wikibooks.org/wiki/..%2FStandalone%20programs%2F`

[2] See the Haskell report for more details on the module system ^{`http://www.haskell.org/onlinereport/modules.html`} .

[3] In Haskell98, the last standardised version of Haskell before Haskell 2010, the module system was fairly conservative, but recent common practice consists of employing a hierarchical module system, using periods to section off namespaces.

```
import Data.Char (toLower, toUpper) -- import only the functions toLower and
 toUpper from Data.Char

import Data.List -- import everything exported from Data.List

import MyModule -- import everything exported from MyModule
```

Imported datatypes are specified by their name, followed by a list of imported constructors in parenthesis. For example:

```
import Data.Tree (Tree(Node)) -- import only the Tree data type and its Node
 constructor from Data.Tree
```

What if you import some modules that have overlapping definitions? Or if you import a module but want to overwrite a function yourself? There are three ways to handle these cases: Qualified imports, hiding definitions, and renaming imports.

22.2.1 Qualified imports

Say MyModule and MyOtherModule both have a definition for `remove_e`, which removes all instances of *e* from a string. However, MyModule only removes lower-case e's, and MyOtherModule removes both upper and lower case. In this case the following code is ambiguous:

```
import MyModule
import MyOtherModule

-- someFunction puts a c in front of the text, and removes all e's from the rest
someFunction :: String -> String
someFunction text = 'c' : remove_e text
```

It isn't clear which `remove_e` is meant! To avoid this, use the **qualified** keyword:

```
import qualified MyModule
import qualified MyOtherModule

someFunction text = 'c' : MyModule.remove_e text -- Will work, removes lower
 case e's
someOtherFunction text = 'c' : MyOtherModule.remove_e text -- Will work, removes
 all e's
someIllegalFunction text = 'c' : remove_e text -- Won't work as there is no
 remove_e defined
```

In the latter code snippet, no function named `remove_e` is available at all. When we do qualified imports, all the imported values include the module names as a prefix. Incidentally, you can also use the same prefixes even if you did a regular import (in our example, `MyModule.remove_e` works even if the "qualified" keyword isn't included).

Note:
There is an ambiguity between a qualified name like `MyModule.remove_e` and the function composition operator `(.)`. Writing `reverse.MyModule.remove_e` is bound to confuse your Haskell compiler. One solution is stylistic: always use spaces for function composition, for example, `reverse . remove_e` or `Just . remove_e` or even `Just . MyModule.remove_e`

22.2.2 Hiding definitions

Now suppose we want to import both MyModule and MyOtherModule, but we know for sure we want to remove all e's, not just the lower cased ones. It will become really tedious to add MyOtherModule before every call to remove_e. Can't we just exclude the remove_e from MyModule?

```
import MyModule hiding (remove_e)
import MyOtherModule

someFunction text = 'c' : remove_e text
```

This works because of the word **hiding** on the import line. Whatever follows the "hiding" keyword will not be imported. Hide multiple items by listing them with parentheses and comma-separation:

```
import MyModule hiding (remove_e, remove_f)
```

Note that algebraic datatypes and type synonyms cannot be hidden. These are always imported. If you have a datatype defined in multiple imported modules, you must use qualified names.

22.2.3 Renaming imports

This is not really a technique to allow for overwriting, but it is often used along with the qualified flag. Imagine:

```
import qualified MyModuleWithAVeryLongModuleName

someFunction text = 'c' : MyModuleWithAVeryLongModuleName.remove_e $ text
```

Especially when using qualified, this gets irritating. We can improve things by using the **as** keyword:

```
import qualified MyModuleWithAVeryLongModuleName as Shorty

someFunction text = 'c' : Shorty.remove_e $ text
```

This allows us to use Shorty instead of MyModuleWithAVeryLongModuleName as prefix for the imported functions. This renaming works with both qualified and regular importing.

As long as there are no conflicting items, we can import multiple modules and rename them the same:

```
import MyModule as My
import MyCompletelyDifferentModule as My
```

In this case, both the functions in MyModule and the functions in MyCompletelyDifferentModule can be prefixed with My.

22.2.4 Combining renaming with limited import

Sometimes it is convenient to use the import directive twice for the same module. A typical scenario is as follows:

```
import qualified Data.Set as Set
import Data.Set (Set, empty, insert)
```

This give access to all of the Data.Set module via the alias ”Set”, and also lets you access a few selected functions (empty, insert, and the constructor) without using the ”Set” prefix.

22.3 Exporting

In the examples at the start of this article, the words ”import *everything exported* from MyModule” were used.[4] This raises a question. How can we decide which functions are exported and which stay ”internal”? Here's how:

```
module MyModule (remove_e, add_two) where

add_one blah = blah + 1

remove_e text = filter (/= 'e') text

add_two blah = add_one . add_one $ blah
```

In this case, only `remove_e` and `add_two` are exported. While `add_two` is allowed to make use of `add_one`, functions in modules that import `MyModule` cannot use `add_one` directly, as it isn't exported.

Datatype export specifications are written similarly to import. You name the type, and follow with the list of constructors in parenthesis:

```
module MyModule2 (Tree(Branch, Leaf)) where

data Tree a = Branch {left, right :: Tree a}
            | Leaf a
```

In this case, the module declaration could be rewritten ”MyModule2 (Tree(..))”, declaring that all constructors are exported.

Maintaining an export list is good practice not only because it reduces namespace pollution but also because it enables certain compile-time optimizations[5] which are unavailable otherwise.

[4] A module may export functions that it imports. Mutually recursive modules are possible but need some special treatment ^{http://www.haskell.org/ghc/docs/latest/html/users_guide/separate-compilation.html#mutual-recursion} .

[5] http://www.haskell.org/haskellwiki/Performance/GHC#Inlining

23 Indentation

Haskell relies on indentation to reduce the verbosity of your code. Despite some complexity in practice, there are really only a couple fundamental layout rules.[1]

23.1 The golden rule of indentation

Code which is part of some expression should be indented further in than the beginning of that expression (even if the expression is not the leftmost element of the line).

The easiest example is a 'let' binding group. The equations binding the variables are part of the 'let' expression, and so should be indented further in than the beginning of the binding group: the 'let' keyword. When you start the expression on a separate line, you only need to indent by one space (although more than one space is also acceptable and may be clearer).

```
let
 x = a
 y = b
```

You may also place the first clause alongside the 'let' as long as you indent the rest to line up:

wrong

```
let x = a
 y = b
```

wrong

```
let x = a
     y = b
```

right

```
let x = a
    y = b
```

This tends to trip up a lot of beginners: All *grouped* expressions must be exactly aligned. On the first line, Haskell counts everything to the left of the expression as indent, even though it is not whitespace.

Here are some more examples:

```
do
  foo
  bar
  baz

do foo
   bar
   baz
```

[1] See section 2.7 of The Haskell Report (lexemes) ^{`http://www.haskell.org/onlinereport/lexemes.html#sect2.7`} on layout.

```
where x = a
      y = b

case x of
  p  -> foo
  p' -> baz
```

Note that with 'case' it is less common to place the first subsidiary expression on the same line as the 'case' keyword (although it would still be valid code). Hence, the subsidiary expressions in a case expression tend to be indented only one step further than the 'case' line. Also note how we lined up the arrows here: this is purely aesthetic and is not counted as different layout; only *indentation* (i.e. whitespace beginning on the far-left edge) makes a difference to the interpretation of the layout.

Things get more complicated when the beginning of an expression is not at the start of a line. In this case, it's safe to just indent further than the line containing the expression's beginning. In the following example, `do` comes at the end of a line, so the subsequent parts of the expression simply need to be indented relative to the line that contains the `do`, not relative to the `do` itself.

```
myFunction firstArgument secondArgument = do
  foo
  bar
  baz
```

Here are some alternative layouts which all work:

```
myFunction firstArgument secondArgument =
  do foo
     bar
     baz

myFunction firstArgument secondArgument = do foo
                                             bar
                                             baz
myFunction firstArgument secondArgument =
  do
     foo
     bar
     baz
```

23.2 Explicit characters in place of indentation

Indentation is actually optional if you instead use semicolons and curly braces for grouping and separation, as in "one-dimensional" languages like C. Even though the consensus among Haskell programmers is that meaningful indentation leads to better-looking code, understanding how to convert from one style to the other can help understand the indentation rules. The entire layout process can be summed up in three translation rules (plus a fourth one that doesn't come up very often):

1. If you see one of the layout keywords, (`let`, `where`, `of`, `do`), insert an open curly brace (right before the stuff that follows it)
2. If you see something indented to the SAME level, insert a semicolon
3. If you see something indented LESS, insert a closing curly brace

4. If you see something unexpected in a list, like **where**, insert a closing brace before instead of a semicolon.

For instance, this definition...

```
foo :: Double -> Double
foo x =
    let s = sin x
        c = cos x
    in 2 * s * c
```

...can be rewritten without caring about the indentation rules as:

```
foo :: Double -> Double;
foo x = let {
  s = sin x;
  c = cos x;
  } in 2 * s * c
```

One circumstance in which explicit braces and semicolons can be convenient is when writing one-liners in GHCi:

```
Prelude> let foo :: Double -> Double; foo x = let { s = sin x; c = cos x } in 2
 * s * c
```

Exercises:
Rewrite this snippet from the Control Structures chapter using explicit braces and semicolons:

```
doGuessing num = do
  putStrLn "Enter your guess:"
  guess <- getLine
  case compare (read guess) num of
    LT -> do putStrLn "Too low!"
             doGuessing num
    GT -> do putStrLn "Too high!"
             doGuessing num
    EQ -> putStrLn "You Win!"
```

23.3 Layout in action

wrong

```
do first thing
second thing
third thing
```

wrong

```
do first thing
 second thing
 third thing
```

right

```
do first thing
   second thing
   third thing
```

right

```
do
  first thing
  second thing
  third thing
```

23.3.1 Indent to the first

Due to the ”golden rule of indentation” described above, a curly brace within a `do` block depends not on the `do` itself but the thing that immediately follows it. For example, this weird-looking block of code is totally acceptable:

```
         do
first thing
second thing
third thing
```

As a result, you could also write combined if/do combination like this:

Wrong

```
if foo
   then do first thing
        second thing
        third thing
   else do something_else
```

Right

```
if foo
   then do first thing
           second thing
           third thing
   else do something_else
```

Right

```
if foo
   then do
     first thing
     second thing
     third thing
   else do
     something_else
```

It isn’t about the `do`, it’s about lining up all the items that are at the same level within the `do`.

Thus, all of the following are acceptable:

```
main = do
  first thing
  second thing
```

or

```
main =
  do
    first thing
    second thing
```

or

```
main =
  do first thing
     second thing
```

23.3.2 `if` within `do`

This is a combination which trips up many Haskell programmers. Why does the following block of code not work?

sweet but wrong

unsweet and wrong

```
-- why is this bad?
do first thing
   if condition
   then foo
   else bar
   third thing
```

```
-- still bad, just explicitly so
do { first thing
   ; if condition
   ; then foo
   ; else bar
   ; third thing }
```

Naturally, the Haskell compiler is confused because it thinks that you never finished writing your `if` expression, before writing a new statement. The compiler sees that you have written something like `if condition;`, which is bad because it is unfinished. In order to fix this, we need to indent the bottom parts of this if block so that `then` and `else` become part of the `if` statement.

sweet and correct

```
-- whew, fixed it!
do first thing
   if condition
     then foo
     else bar
   third thing
```

unsweet and correct

```
-- the fixed version without sugar
do { first thing
   ; if condition
      then foo
      else bar
   ; third thing }
```

Now, the do block sees the whole if statement as one item. When if-then-else statements are not within do blocks, this specific indentation isn't technically necessary, but it never hurts, so it is a good habit to always indent if-then-else in this way.

Exercises:
The if-within-do issue has tripped up so many Haskellers that one programmer has posted a proposal[a] to the Haskell prime initiative to add optional semicolons between `if then else`. How would that help?

[a] `http://hackage.haskell.org/trac/haskell-prime/ticket/23`

Issues with indentation are explained further in connection with showing how `do` is syntactic sugar for the monadic operator `(>>=)`. See Translating the bind operator[2] and the associated footnote about indentation[3].

[2] Chapter 34 on page 199

[3] Chapter 34 on page 199

24 More on datatypes

24.1 Enumerations

One special case of the `data` declaration is the *enumeration* — a data type where none of the constructor functions have any arguments:

```
data Month = January | February | March | April | May | June | July
           | August | September | October | November | December
```

You can mix constructors that do and do not have arguments, but then the result is not called an enumeration. The following example is *not* an enumeration because the last constructor takes three arguments:

```
data Colour = Black | Red | Green | Blue | Cyan
            | Yellow | Magenta | White | RGB Int Int Int
```

As you will see further on when we discuss classes and derivation, there are practical reasons to distinguish between what is and isn't an enumeration.

Incidentally, the `Bool` datatype is an enumeration:

```
data Bool = False | True
    deriving (Bounded, Enum, Eq, Ord, Read, Show)
```

24.2 Named Fields (Record Syntax)

Consider a datatype whose purpose is to hold configuration settings. Usually, when you extract members from this type, you really only care about one or two of the many settings. Moreover, if many of the settings have the same type, you might often find yourself wondering "wait, was this the fourth or *fifth* element?" One way to clarify is to write accessor functions. Consider the following made-up configuration type for a terminal program:

```
data Configuration = Configuration
    String   -- User name
    String   -- Local host
    String   -- Remote host
    Bool     -- Is guest?
    Bool     -- Is superuser?
    String   -- Current directory
    String   -- Home directory
    Integer  -- Time connected
  deriving (Eq, Show)
```

You could then write accessor functions, such as:

```
getUserName (Configuration un _ _ _ _ _ _ _) = un
getLocalHost (Configuration _ lh _ _ _ _ _ _) = lh
getRemoteHost (Configuration _ _ rh _ _ _ _ _) = rh
getIsGuest (Configuration _ _ _ ig _ _ _ _) = ig
-- And so on...
```

You could also write update functions to update a single element. Of course, if you add or remove an element in the configuration later, all of these functions now have to take a different number of arguments. This is quite annoying and is an easy place for bugs to slip in. Thankfully, there's a solution: we simply give names to the fields in the datatype declaration, as follows:

```
data Configuration = Configuration
    { username      :: String
    , localHost     :: String
    , remoteHost    :: String
    , isGuest       :: Bool
    , isSuperuser   :: Bool
    , currentDir    :: String
    , homeDir       :: String
    , timeConnected :: Integer
    }
```

This will automatically generate the following accessor functions for us:

```
username :: Configuration -> String
localHost :: Configuration -> String
-- etc.
```

This also gives us a convenient update method. Here is a short example for a ”post working directory” and ”change directory” functions that work on `Configurations`:

```
changeDir :: Configuration -> String -> Configuration
changeDir cfg newDir =
    if directoryExists newDir -- make sure the directory exists
        then cfg { currentDir = newDir }
        else error "Directory does not exist"

postWorkingDir :: Configuration -> String
postWorkingDir cfg = currentDir cfg
```

So, in general, to update the field `x` in a datatype `y` to `z`, you write `y { x = z }`. You can change more than one; each should be separated by commas, for instance, `y {x = z, a = b, c = d }`.

Note:
Those of you familiar with object-oriented languages might be tempted, after all of this talk about ”accessor functions” and ”update methods”, to think of the `y{x=z}` construct as a setter method, which modifies the value of x in a pre-existing y. It is **not** like that – remember that in Haskell variables are immutable[a]. Therefore, using the example above, if you do something like `conf2 = changeDir conf1 "/opt/foo/bar"` `conf2` will be defined as a `Configuration` which is just like `conf1` except for having `"/opt/foo/bar"` as its `currentDir`, but `conf1` will remain unchanged.

[a] Chapter 3.4 on page 9

24.2.1 It's only sugar

You can, of course, continue to pattern match against Configurations as you did before. The named fields are simply syntactic sugar; you can still write something like:

```
getUserName (Configuration un _ _ _ _ _ _ _) = un
```

But there is no need to do this.

Finally, you can pattern match against named fields as in:

```
getHostData (Configuration { localHost = lh, remoteHost = rh }) = (lh, rh)
```

This matches the variable lh against the localHost field in the Configuration and the variable rh against the remoteHost field. These matches will succeed, of course. You could also constrain the matches by putting values instead of variable names in these positions, as you would for standard datatypes.

If you are using GHC, then, with the language extension NamedFieldPuns, it is also possible to use this form:

```
getHostData (Configuration { localHost, remoteHost }) = (localHost, remoteHost)
```

It can be mixed with the normal form like this:

```
getHostData (Configuration { localHost, remoteHost = rh }) = (localHost, rh)
```

(To use this language extension, enter :set -XNamedFieldPuns in the interpreter, or use the {-# LANGUAGE NamedFieldPuns #-} pragma at the beginning of a source file, or pass the -XNamedFieldPuns command-line flag to the compiler.)

You can create values of Configuration in the old way as shown in the first definition below, or in the named field's type, as shown in the second definition:

```
initCFG = Configuration "nobody" "nowhere" "nowhere" False False "/" "/" 0

initCFG' = Configuration
    { username      = "nobody"
    , localHost     = "nowhere"
    , remoteHost    = "nowhere"
    , isguest       = False
    , issuperuser   = False
    , currentdir    = "/"
    , homedir       = "/"
    , timeConnected = 0
    }
```

The first way is much shorter, but the second is much clearer.

WARNING: The second style will allow you to write code that omits fields but will still compile, such as:

```
cfgFoo = Configuration { username = "Foo" }
cfgBar = Configuration { localHost = "Bar", remoteHost = "Baz" }
cfgUndef = Configuration {}
```

Trying to evaluate the unspecified fields will then result in a runtime error!

24.3 Parameterized Types

Parameterized types are similar to "generic" or "template" types in other languages. A parameterized type takes one or more type parameters. For example, the Standard Prelude type `Maybe` is defined as follows:

```
data Maybe a = Nothing | Just a
```

This says that the type `Maybe` takes a type parameter `a`. You can use this to declare, for example:

```
lookupBirthday :: [Anniversary] -> String -> Maybe Anniversary
```

The `lookupBirthday` function takes a list of birthday records and a string and returns a `Maybe Anniversary`. The usual interpretation of such a type is that if the name given through the string is found in the list of anniversaries the result will be `Just` the corresponding record; otherwise, it will be `Nothing`. `Maybe` is the simplest and most common way of indicating failure in Haskell. It is also sometimes seen in the types of function arguments, as a way to make them optional (the intent being that passing `Nothing` amounts to omitting the argument).

You can parameterize `type` and `newtype` declarations in exactly the same way. Furthermore you can combine parameterized types in arbitrary ways to construct new types.

24.3.1 More than one type parameter

We can also have more than one type parameter. An example of this is the `Either` type:

```
data Either a b = Left a | Right b
```

For example:

```
pairOff :: Int -> Either String Int
pairOff people
    | people < 0  = Left "Can't pair off negative number of people."
    | people > 30 = Left "Too many people for this activity."
    | even people = Right (people `div` 2)
    | otherwise   = Left "Can't pair off an odd number of people."

groupPeople :: Int -> String
groupPeople people =
    case pairOff people of
        Right groups -> "We have " ++ show groups ++ " group(s)."
        Left problem -> "Problem! " ++ problem
```

In this example `pairOff` indicates how many groups you would have if you paired off a certain number of people for your activity. It can also let you know if you have too many people for your activity or if somebody will be left out. So `pairOff` will return either an Int representing the number of groups you will have, or a String describing the reason why you can't create your groups.

24.3.2 Kind Errors

The flexibility of Haskell parameterized types can lead to errors in type declarations that are somewhat like type errors, except that they occur in the type declarations rather than in the program proper. Errors in these ”types of types” are known as ”kind” errors. You don’t program with kinds: the compiler infers them for itself. But if you get parameterized types wrong then the compiler will report a kind error.

25 Other data structures

In this chapter, we will work through examples of how the techniques we have studied thus far can be used to deal with more complex data types. In particular, we will see examples of *recursive data structures*, which are data types that can contain values of the same type. Recursive data structures play a vital role in many programming techniques, and so even if you are not going to need defining a new one often (as opposed to using the ones available from the libraries) it is important to be aware of what they are and how they can be manipulated. Besides that, following closely the implementations in this chapter is a good exercise for your budding Haskell abilities.

Note:
The Haskell library ecosystem provides a wealth of data structures (recursive and otherwise), covering a wide range of practical needs. Beyond lists, there are maps, sets, finite sequences and arrays, among many others. A good place to begin learning about the main ones is the Data structures primer[a] in the Haskell in Practice track. We recommend you to at least skim it once you finish the next few Intermediate Haskell chapters.

[a] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FData%20structures%20primer

25.1 Trees

One of the most important types of recursive data structures are *trees.* There are several different kinds of trees, so we will arbitrarily choose a simple one to use as an example. Here is its definition:

```
data Tree a = Leaf a | Branch (Tree a) (Tree a)
```

As you can see, it's parameterized; i.e. we can have trees of `Int`s, trees of `String`s, trees of `Maybe Int`s, trees of `(Int, String)` pairs and so forth. What makes this data type special is that `Tree` appears in the definition of itself. A `Tree a` is either a *leaf*, containing a value of type `a` or a *branch*, from which hang two other trees of type `Tree a`.

25.1.1 Lists as Trees

As we have seen in More about lists[1] and List Processing[2], we break lists down into two cases: An empty list (denoted by `[]`), and an element of the specified type plus another list (denoted by `(x:xs)`). That means the definition of the list data type might look like this:

[1] https://en.wikibooks.org/wiki/Haskell%2FMore%20about%20lists
[2] https://en.wikibooks.org/wiki/Haskell%2FList_Processing

```
-- Pseudo-Haskell, will not actually work (because lists have special syntax).
data [a] = [] | (a:[a])
```

An equivalent definition you can actually play with is:

```
data List a = Nil | Cons a (List a)
```

Like trees, lists are also recursive. For lists, the constructor functions are `[]` and `(:)`. They correspond to `Leaf` and `Branch` in the `Tree` definition above. That implies we can use `Leaf` and `Branch` for pattern matching just as we did with the empty list and the `(x:xs)`.

25.1.2 Maps and Folds

We already know about maps and folds for lists. Now, we can write map and fold functions for our new `Tree` type. To recap:

```
data Tree a = Leaf a | Branch (Tree a) (Tree a) deriving (Show)
data [a]    = []     | (:)    a [a]
  -- (:) a [a] is the same as (a:[a]) with prefix instead of infix notation.
```

> **Note:**
> Deriving is explained later on in the section Class Declarations[a]. For now, understand it as telling Haskell (and by extension your interpreter) how to display a Tree instance.

[a] Chapter 26 on page 155

Map

Let's take a look at the definition of `map` for lists:

```
map :: (a -> b) -> [a] -> [b]
map _ [] = []
map f (x:xs) = f x : map f xs
```

If we were to write `treeMap`, what would its type be? Defining the function is easier if you have an idea of what its type should be.

We want `treeMap` to work on a `Tree` of some type and return another `Tree` of some type by applying a function on each element of the tree.

```
treeMap :: (a -> b) -> Tree a -> Tree b
```

This is just like the list example.

Now, when talking about a `Tree`, each `Leaf` only contains a single value, so all we have to do is apply the given function to that value and then return a `Leaf` with the altered value:

```
treeMap :: (a -> b) -> Tree a -> Tree b
treeMap f (Leaf x) = Leaf (f x)
```

This looks a lot like the empty list case with `map`. Now, if we have a `Branch`, it will include two subtrees; what do we do with those? The list-`map` uses a call to itself on the tail of the list, so we also shall do that with the two subtrees. The complete definition of `treeMap` is as follows:

```
treeMap :: (a -> b) -> Tree a -> Tree b
treeMap f (Leaf x) = Leaf (f x)
treeMap f (Branch left right) = Branch (treeMap f left) (treeMap f right)
```

We can make this a bit more readable by noting that `treeMap f` is itself a function with type `Tree a -> Tree b`. This gives us the following revised definition:

```
treeMap :: (a -> b) -> Tree a -> Tree b
treeMap f = g where
  g (Leaf x) = Leaf (f x)
  g (Branch left right) = Branch (g left) (g right)
```

If you didn't follow that immediately, try re-reading it. This use of pattern matching may seem weird at first, but it is essential to the use of datatypes. Remember that pattern matching happens on constructor functions.

When you're ready, read on for folds over Trees.

Fold

As with map, let's first review the definition of `foldr` for lists:

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr f acc [] = acc
foldr f acc (x:xs) = f x (foldr f acc xs)
```

Recall that lists have two constructors:

```
(:) :: a -> [a] -> [a]  -- takes an element and combines it with the rest of the
 list
[] :: [a]  -- the empty list takes zero arguments
```

Thus `foldr` takes two arguments corresponding to the two constructors:

```
f :: a -> b -> b  -- a function takes two elements and operates on them to
 return a single result
acc :: b  -- the accumulator defines what happens with the empty list
```

Let's take a moment to make this clear. If the initial `foldr` is given an empty list, then the default accumulator is returned. For functions like `(+)`, the initial accumulator will be 0. With a non-empty list, the value returned by each fold is used in the next fold. When the list runs out, we are back at the empty list, so foldr returns whatever is *then* the accumulator value from the last fold.

Like `foldr` for lists, we want `treeFold` to transform a tree of some type into a value of some other type; so in place of `[a] -> b` we will have `Tree a -> b`. How do we specify the transformation? First note that `Tree a` has two constructors (just like lists have two constructors):

```
Branch :: Tree a -> Tree a -> Tree a
Leaf :: a -> Tree a
```

So treeFold will have two arguments corresponding to the two constructors:

```
fbranch :: b -> b -> b
fleaf :: a -> b
```

Putting it all together we get the following type definition:

```
treeFold :: (b -> b -> b) -> (a -> b) -> Tree a -> b
```

That is, the first argument, of type (b -> b -> b), is a function specifying how to combine subtrees into a single result; the second argument, of type a -> b, is a function specifying what to do with leaves (which are the end of recursion, just like empty-list for lists); and the third argument, of type Tree a, is the whole tree we want to fold.

As with treeMap, we'll avoid repeating the arguments fbranch and fleaf by introducing a local function g:

```
treeFold :: (b -> b -> b) -> (a -> b)  -> Tree a -> b
treeFold fbranch fleaf = g where
  -- definition of g goes here
```

The argument fleaf tells us what to do with Leaf subtrees:

```
g (Leaf x) = fleaf x
```

The argument fbranch tells us how to combine the results of "folding" two subtrees:

```
g (Branch left right) = fbranch (g left) (g right)
```

Our full definition becomes:

```
treeFold :: (b -> b -> b) -> (a -> b) -> Tree a -> b
treeFold fbranch fleaf = g where
  g (Leaf x) = fleaf x
  g (Branch left right) = fbranch (g left) (g right)
```

For examples of how these work, copy the Tree data definition and the treeMap and treeFold functions to a Haskell file, along with the following example Tree and example functions to fold over.

```
tree1 :: Tree Integer
tree1 =
    Branch
       (Branch
           (Branch
               (Leaf 1)
               (Branch (Leaf 2) (Leaf 3)))
           (Branch
               (Leaf 4)
               (Branch (Leaf 5) (Leaf 6))))
       (Branch
           (Branch (Leaf 7) (Leaf 8))
           (Leaf 9))

doubleTree = treeMap (*2)  -- doubles each value in tree
```

```
sumTree = treeFold (+) id -- sum of the leaf values in tree
fringeTree = treeFold (++) (: [])  -- list of the leaves of tree
```

Then load it into GHCi and evaluate:

```
doubleTree tree1
sumTree tree1
fringeTree tree1
```

25.2 Other datatypes

Map and fold functions can be defined for any kind of data type. In order to generalize the strategy applied for lists and trees, in this final section we will work out a map and a fold for a very strange, intentionally-contrived datatype:

```
data Weird a b = First a
               | Second b
               | Third [(a,b)]
               | Fourth (Weird a b)
```

It can be a useful exercise to write the functions as you follow the examples, trying to keep the coding one step ahead of your reading.

25.2.1 General Map

The first important difference in working with this `Weird` type is that it has *two* type parameters. For that reason, we will want the map function to take two functions as arguments, one to be applied on the elements of type `a` and another for the elements of type `b`. With that accounted for, we can write the type signature of `weirdMap`:

```
weirdMap :: (a -> c) -> (b -> d) -> Weird a b -> Weird c d
```

Next step is defining `weirdMap`. The key point is that maps preserve the *structure* of a datatype, so the function must evaluate to a `Weird` which uses the same constructor as the one used for the original `Weird`. For that reason, we need one definition to handle each constructor, and these constructors are used as patterns for writing them. As before, to avoid repeating the `weirdMap` argument list over and over again a **where** clause comes in handy. A sketch of the function is below:

```
weirdMap :: (a -> c) -> (b -> d) -> Weird a b -> Weird c d
weirdMap fa fb = g
  where
    g (First x)          = --More to follow
    g (Second y)         = --More to follow
    g (Third z)          = --More to follow
    g (Fourth w)         = --More to follow
```

The first two cases are fairly straightforward, as there is just a single element of `a` or `b` type inside the `Weird`.

```
weirdMap :: (a -> c) -> (b -> d) -> Weird a b -> Weird c d
weirdMap fa fb = g
  where
    g (First x)          = First (fa x)
    g (Second y)         = Second (fb y)
    g (Third z)          = --More to follow
    g (Fourth w)         = --More to follow
```

`Third` is trickier because it contains a list whose elements are themselves data structures (the tuples). So we need to navigate the nested data structures, apply `fa` and `fb` on all elements of type `a` and `b` and eventually (as a map must preserve structure) produce a list of tuples – `[(c,d)]` – to be used with the constructor. The simplest approach might seem to be just breaking down the list inside the `Weird` and playing with the patterns:

```
    g (Third []) = Third []
    g (Third ((x,y):zs)) = Third ( (fa x, fb y) : ( (\(Third z) -> z) (g (Third
 zs)) ) )
```

This appears to be written as a typical recursive function for lists. We start by applying the functions of interest to the first element in order to obtain the head of the new list, `(fa x, fb y)`. But what will we cons it to? As `g` requires a `Weird` argument, we need to make a `Weird` using the list tail in order to make the recursive call. But then `g` will give a `Weird` and not a list, so we have to retrieve the modified list from that – that's the role of the lambda function. Finally, there is also the empty list base case to be defined as well.

After all of that, we are left with a messy function. Every recursive call of `g` requires wrapping `zs` into a `Weird`, while what we really wanted to do was to build a list with `(fa x, fb y)` and the modified `xs`. The problem with this solution is that `g` can (thanks to pattern matching) act directly on the list head but (due to its type signature) can't be called directly on the list tail. For that reason, it would be better to apply `fa` and `fb` without breaking down the list with pattern matching (as far as `g` is directly concerned, at least). But there *was* a way to directly modify a list element-by-element...

```
    g (Third z) = Third ( map (\(x, y) -> (fa x, fb y) ) z)
```

...our good old `map` function, which modifies all tuples in the list `z` using a lambda function. In fact, the first attempt at writing the definition looked just like an application of the list map except for the spurious `Weird` packing and unpacking. We got rid of these by having the pattern splitting of `z` done by `map`, which works directly with regular lists. You could find it useful to expand the map definition inside `g` to see the difference more clearly. Finally, you may prefer to write this new version in an alternative and clean way using list comprehension syntax:

```
    g (Third z) = Third [ (fa x, fb y) | (x,y) <- z ]
```

Adding the `Third` function, we only have the `Fourth` left to define:

```
weirdMap :: (a -> c) -> (b -> d) -> Weird a b -> Weird c d
weirdMap fa fb = g
  where
    g (First x)          = First (fa x)
    g (Second y)         = Second (fb y)
    g (Third z)          = Third ( map (\(x, y) -> (fa x, fb y) ) z)
    g (Fourth w)         = --More to follow
```

All we need to do is apply `g` recursively:

```
weirdMap :: (a -> c) -> (b -> d) -> Weird a b -> Weird c d
weirdMap fa fb = g
  where
    g (First x)          = First (fa x)
    g (Second y)         = Second (fb y)
    g (Third z)          = Third ( map (\(x, y) -> (fa x, fb y) ) z)
    g (Fourth w)         = Fourth (g w)
```

25.2.2 General Fold

While we were able to define a map by specifying as arguments a function for every separate type, this isn't enough for a fold. For a fold, we'll need a function for every constructor function. With lists, the constructors are `[]` and `(:)`. The `acc` argument in the `foldr` function corresponds to the `[]` constructor. The `f` argument in the `foldr` function corresponds to the `(:)` constructor. The `Weird` datatype has four constructors, so we need four functions – one for handling the internal structure of the datatype specified by each constructor. Next, we have an argument of the `Weird a b` type, and finally we want the whole fold function to evaluate to a value of some other, arbitrary, type. Additionally, each individual function we pass to `weirdFold` must evaluate to the same type `weirdFold` does. That allows us to make a mock type signature and sketch the definition:

```
weirdFold :: (something1 -> c) -> (something2 -> c) -> (something3 -> c) ->
 (something4 -> c) -> Weird a b -> c
weirdFold f1 f2 f3 f4 = g
  where
    g (First x)          = --Something of type c here
    g (Second y)         = --Something of type c here
    g (Third z)          = --Something of type c here
    g (Fourth w)         = --Something of type c here
```

Now, we need to figure out to which types `something1`, `something2`, `something3` and `something4` correspond to. That is done by analyzing the constructors, since the functions must take as arguments the elements of the datatype (whose types are specified by the constructor type signature). Again, the types and definitions of the first two functions are easy enough. The third one isn't too difficult either because, for the purposes of folding the list of `(a,b)`, tuples are no different from a simple type (unlike in the map example, the *internal* structure does not concern us now). The fourth constructor, however, is recursive, and we have to be careful. As with `weirdMap`, we also need to recursively call the `g` function. This brings us to the final definition:

```
weirdFold :: (a -> c) -> (b -> c) -> ([(a,b)] -> c) -> (c -> c) -> Weird a b ->
 c
weirdFold f1 f2 f3 f4 = g
  where
    g (First x)          = f1 x
    g (Second y)         = f2 y
    g (Third z)          = f3 z
    g (Fourth w)         = f4 (g w)
```

Note:
If you were expecting very complex expressions in the `weirdFold` above and are surprised by the immediacy of the solution, it might be helpful to have a look on what the common `foldr` would look like if we wrote it in this style and didn't have the special square-bracket syntax of lists to distract us:

```
-- List a is [a], Cons is (:) and Nil is []
data List a = Cons a (List a) | Nil

listFoldr :: (a -> b -> b) -> (b) -> List a -> b
listFoldr fCons fNil = g
  where
    g (Cons x xs) = fCons x (g xs)
    g Nil         = fNil
```

Now it is easier to see the parallels. The extra complications are that `Cons` (that is, `(:)`) takes two arguments (and, for that reason, so does `fCons`) and is recursive, requiring a call to `g`. Also, `fNil` is of course not really a function, as it takes no arguments.

Folds on recursive datatypes

As far as folds are concerned, `Weird` was a fairly nice datatype to deal with. Just one recursive constructor, which isn't even nested inside other structures. What would happen if we added a truly complicated fifth constructor?

```
    Fifth [Weird a b] a (Weird a a, Maybe (Weird a b))
```

This is a valid and yet tricky question. In general, the following rules apply:

- A function to be supplied to a fold has the same number of arguments as the corresponding constructor.
- The type of the arguments of such a function match the types of the constructor arguments, *except* if the constructor is recursive (that is, takes an argument of its own type).
- If a constructor is recursive, any recursive argument of the constructor will correspond to an argument of the type the fold evaluates to.[3]
- If a constructor is recursive, the complete fold function should be (recursively) applied to the recursive constructor arguments.
- If a recursive element appears inside another data structure, the appropriate map function for that data structure should be used to apply the fold function to it.

So `f5` would have the type:

```
f5 :: [c] -> a -> (Weird a a, Maybe c) -> c
```

[3] This sort of recursiveness, in which the function used for folding can take the result of another fold as an argument, is what confers the folds of data structures such as lists and trees their ”accumulating” functionality.

as the type of Fifth is:

```
Fifth :: [Weird a b] -> a -> (Weird a a, Maybe (Weird a b)) -> Weird a b
```

The definition of g for the Fifth constructor will be:

```
g (Fifth list x (waa, mc)) = f5 (map g list) x (waa, maybeMap g mc)
  where
    maybeMap f Nothing = Nothing
    maybeMap f (Just w) = Just (f w)
```

Note that nothing strange happens with the Weird a a part. No g gets called. What's up? This is recursion, right? Well, not really. Weird a a and Weird a b are different types, so it isn't a real recursion. It isn't guaranteed that, for example, f2 will work with something of type 'a', where it expects a type 'b'. It can be true for some cases but is not reliable for every case.

Also look at the definition of maybeMap. Verify that it is indeed a map function as:

- It preserves structure.
- Only types are changed.

A nice sounding word

The folds we have defined here are examples of *catamorphisms*. A catamorphism is a general way to collapse a data structure into a single value. There is deep theory associated with catamorphisms and related recursion schemes; however, we won't go through any of it now, as our main goal here was exercising the mechanics of data structure manipulation in Haskell with believable examples.

26 Classes and types

Back in Type basics II[1] we had a brief encounter with type classes as the mechanism used with number types. As we hinted back then, however, classes have many other uses.

Broadly speaking, the point of type classes is to ensure that certain operations will be available for values of chosen types. For example, if we know a type belongs to (or, to use the jargon, *instantiates*) the class `Fractional`, then we are guaranteed to, among other things, be able to perform real division with its values.[2]

26.1 Classes and instances

Up to now we have seen how existing type classes appear in signatures such as:

```
(==) :: (Eq a) => a -> a -> Bool
```

Now it is time to switch perspectives. First, we quote the definition of the `Eq` class from Prelude:

```
class  Eq a  where
   (==), (/=) :: a -> a -> Bool

       -- Minimal complete definition:
       --       (==) or (/=)
   x /= y     =  not (x == y)
   x == y     =  not (x /= y)
```

The definition states that if a type `a` is to be made an *instance* of the class `Eq` it must support the functions `(==)` and `(/=)` - the *class methods* - both of them having type `a -> a -> Bool`. Additionally, the class provides default definitions for `(==)` and `(/=)` *in terms of each other*. As a consequence, there is no need for a type in `Eq` to provide both definitions - given one of them, the other will be generated automatically.

With a class defined, we proceed to make existing types instances of it. Here is an arbitrary example of an algebraic data type made into an instance of `Eq` by an *instance declaration*:

```
data Foo = Foo {x :: Integer, str :: String}

instance Eq Foo where
   (Foo x1 str1) == (Foo x2 str2) = (x1 == x2) && (str1 == str2)
```

[1] Chapter 7 on page 37

[2] *To programmers coming from object-oriented languages*: A class in Haskell in all likelihood is *not* what you expect - don't let the terms confuse you. While some of the uses of type classes resemble what is done with abstract classes or Java interfaces, there are fundamental differences which will become clear as we advance.

And now we can apply `(==)` and `(/=)` to `Foo` values in the usual way:

```
• Main> Foo 3 "orange" == Foo 6 "apple" False
• Main> Foo 3 "orange" /= Foo 6 "apple" True
```

A few important remarks:

- The class `Eq` is defined in the Standard Prelude. This code sample defines the type `Foo` and then declares it to be an instance of `Eq`. The three definitions (class, data type, and instance) are *completely separate* and there is no rule about how they are grouped. This works both ways: you could just as easily create a new class `Bar` and then declare the type `Integer` to be an instance of it.
- *Classes are not types*, but categories of types; and so the instances of a class are types instead of values.[3]
- The definition of `(==)` for `Foo` relies on the fact that the values of its fields (namely `Integer` and `String`) are also members of `Eq`. In fact, almost all types in Haskell are members of `Eq` (the most notable exception being functions).
- Type synonyms defined with `type` keyword cannot be made instances of a class.

26.2 Deriving

Since equality tests between values are commonplace, in all likelihood most of the data types you create in any real program should be members of `Eq`. A lot of them will also be members of other Prelude classes such as `Ord` and `Show`. To avoid large amounts of boilerplate for every new type, Haskell has a convenient way to declare the ”obvious” instance definitions using the keyword `deriving`. So, `Foo` would be written as:

```
data Foo = Foo {x :: Integer, str :: String}
    deriving (Eq, Ord, Show)
```

This makes `Foo` an instance of `Eq` with an automatically generated definition of `==` exactly equivalent to the one we just wrote, and also makes it an instance of `Ord` and `Show` for good measure.

You can only use `deriving` with a limited set of built-in classes, which are described *very* briefly below:

Eq

Equality operators `==` and `/=`

Ord

Comparison operators `< <= > >=`; `min`, `max`, and `compare`.

Enum

For enumerations only. Allows the use of list syntax such as `[Blue .. Green]`.

[3] This is a key difference from most OO languages, where a class is also itself a type.

Bounded

Also for enumerations, but can also be used on types that have only one constructor. Provides `minBound` and `maxBound` as the lowest and highest values that the type can take.

Show

Defines the function `show`, which converts a value into a string, and other related functions.

Read

Defines the function `read`, which parses a string into a value of the type, and other related functions.

The precise rules for deriving the relevant functions are given in the language report. However, they can generally be relied upon to be the ”right thing” for most cases. The types of elements inside the data type must also be instances of the class you are deriving.

This provision of special ”magic” function synthesis for a limited set of predefined classes goes against the general Haskell philosophy that ”built in things are not special”, but it does save a lot of typing. Besides that, deriving instances stops us from writing them in the wrong way (an example: an instance of `Eq` such that `x == y` would not be equal to `y == x` would be flat out wrong). [4]

26.3 Class inheritance

Classes can inherit from other classes. For example, here is the main part of the definition of `Ord` in Prelude:

```
class  (Eq a) => Ord a  where
    compare              :: a -> a -> Ordering
    (<), (<=), (>=), (>) :: a -> a -> Bool
    max, min             :: a -> a -> a
```

The actual definition is rather longer and includes default implementations for most of the functions. The point here is that `Ord` inherits from `Eq`. This is indicated by the `=>` notation in the first line, which mirrors the way classes appear in type signatures. Here, it means that for a type to be an instance of `Ord` it must also be an instance of `Eq`, and hence needs to implement the `==` and `/=` operations.[5]

A class can inherit from several other classes: just put all of its superclasses in the parentheses before the `=>`. Let us illustrate that with yet another Prelude quote:

```
class  (Num a, Ord a) => Real a  where
    -- | the rational equivalent of its real argument with full precision
    toRational          ::  a -> Rational
```

[4] There are ways to make the magic apply to other classes. GHC extensions allow `deriving` for a few other common classes for which there is only one correct way of writing the instances, and the GHC generics machinery make it possible to generate instances automatically for custom classes.

[5] If you check the full definition in the Prelude ^{`http://hackage.haskell.org/packages/archive/base/4.1.0.0/doc/html/Prelude.html`} specification, the reason for that becomes clear: the default implementations involve applying `(==)` to the values being compared.

26.4 Standard classes

This diagram, copied from the Haskell Report, shows the relationships between the classes and types in the Standard Prelude. The names in bold are the classes, while the non-bold text stands for the types that are instances of each class (`(->)` refers to functions and `[]`, to lists). The arrows linking classes indicate the inheritance relationships, pointing to the inheriting class.

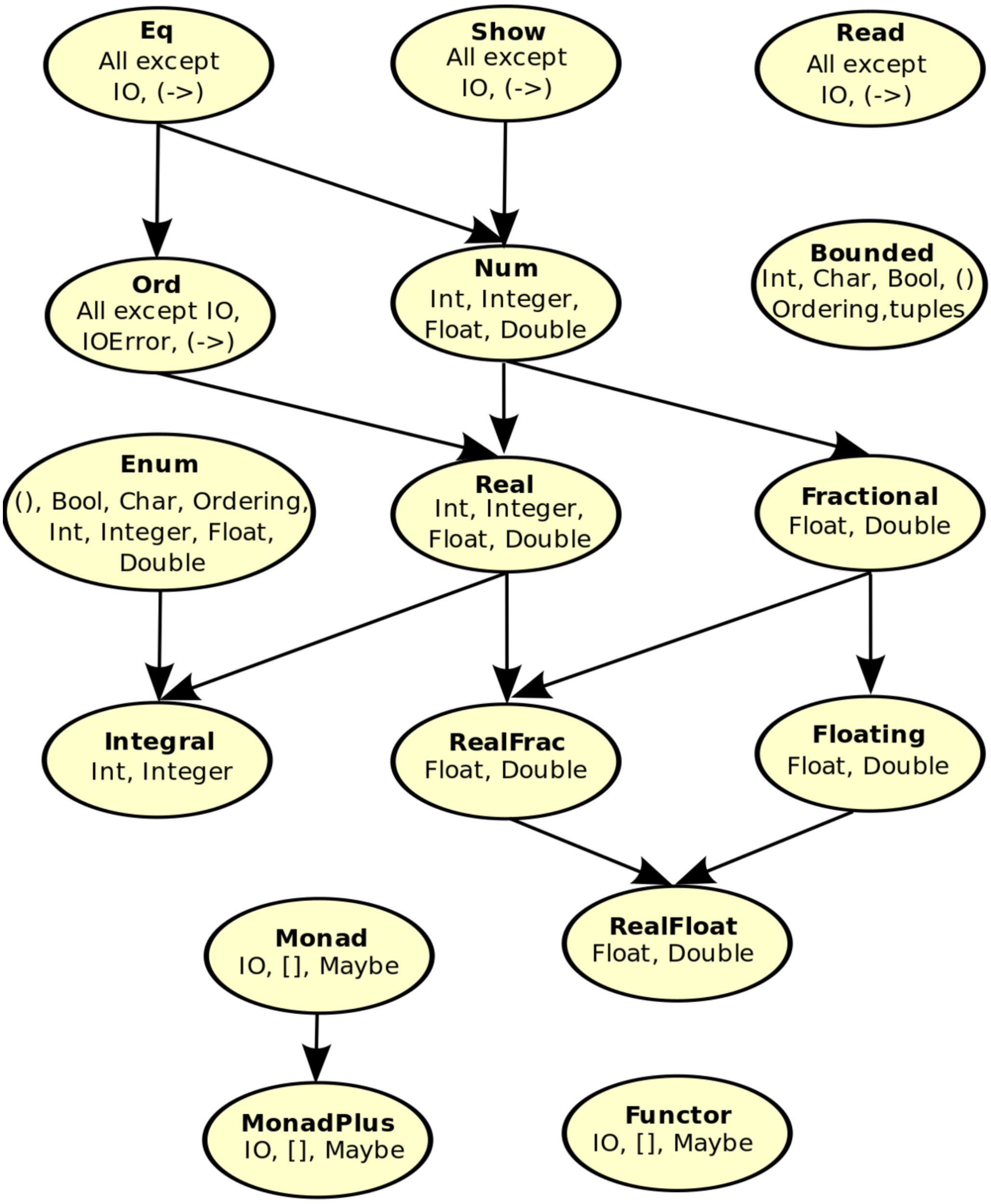


Figure 1

26.5 Type constraints

With all pieces in place, we can go full circle by returning to the very first example involving classes in this book:

```
(+) :: (Num a) => a -> a -> a
```

`(Num a) =>` is a *type constraint*, which restricts the type `a` to instances of the class `Num`. In fact, `(+)` is a method of `Num`, along with quite a few other functions (notably, `(*)` and `(-)`; but not `(/)`).

You can put several limits into a type signature like this:

```
foo :: (Num a, Show a, Show b) => a -> a -> b -> String
foo x y t =
   show x ++ " plus " ++ show y ++ " is " ++ show (x+y) ++ ".  " ++ show t
```

Here, the arguments `x` and `y` must be of the same type, and that type must be an instance of both `Num` and `Show`. Furthermore, the final argument `t` must be of some (possibly different) type that is also an instance of `Show`. This example also displays clearly how constraints propagate from the functions used in a definition (in this case, `(+)` and `show`) to the function being defined.

26.5.1 Other uses

Beyond simple type signatures, type constraints can be introduced in a number of other places:

- `instance` declarations (typical with parametrized types);
- `class` declarations (constraints can be introduced in the method signatures in the usual way for any type variable other than the one defining the class[6]);
- `data` declarations,[7] where they act as constraints for the constructor signatures.

[6] Constraints for the type defining the class should be set via class inheritance.

[7] And `newtype` declarations as well, but not `type`.

Note:
Type constraints in `data` declarations are less useful than it might seem at first. Consider:

```
data (Num a) => Foo a = F1 a | F2 a String
```

Here, `Foo` is a type with two constructors, both taking an argument of a type `a` which must be in `Num`. However, the `(Num a) =>` constraint is only effective for the `F1` and `F2` constructors, and not for other functions involving `Foo`. Therefore, in the following example...

```
fooSquared :: (Num a) => Foo a -> Foo a
fooSquared (F1 x)   = F1 (x * x)
fooSquared (F2 x s) = F2 (x * x) s
```

... even though the constructors ensure `a` will be some type in `Num` we can't avoid duplicating the constraint in the signature of `fooSquared`.[a]

[a] *Extra note for the curious*: This issue is related to some of the problems tackled by the advanced features discussed in the "Fun with types" chapter of the Advanced Track.

26.6 A concerted example

To provide a better view of the interplay between types, classes, and constraints, we will present a very simple and somewhat contrived example. We will define a `Located` class, a `Movable` class which inherits from it, and a function with a `Movable` constraint implemented using the methods of the parent class, i.e. `Located`.

```
-- Location, in two dimensions.
class Located a where
    getLocation :: a -> (Int, Int)

class (Located a) => Movable a where
    setLocation :: (Int, Int) -> a -> a

-- An example type, with accompanying instances.
data NamedPoint = NamedPoint
    { pointName :: String
    , pointX    :: Int
    , pointY    :: Int
    } deriving (Show)

instance Located NamedPoint where
    getLocation p = (pointX p, pointY p)

instance Movable NamedPoint where
    setLocation (x, y) p = p { pointX = x, pointY = y }

-- Moves a value of a Movable type by the specified displacement.
-- This works for any movable, including NamedPoint.
move :: (Movable a) => (Int, Int) -> a -> a
move (dx, dy) p = setLocation (x + dx, y + dy) p
    where
    (x, y) = getLocation p
```

26.6.1 A word of advice

Do not read too much into the `Movable` example just above; it is merely a demonstration of class-related language features. It would be a mistake to think that every single functionality which might be conceivably generalized, such as `setLocation`, needs a type class of its own. In particular, if all your `Located` instances should be able to be moved as well then `Movable` is unnecessary - and if there is just one instance there is no need for type classes at all! Classes are best used when there are several types instantiating it (or if you expect others to write additional instances) and you do not want users to know or care about the differences between the types. An extreme example would be `Show`: general-purpose functionality implemented by an immense number of types, about which you do not need to know a thing about before calling `show`. In the following chapters, we will explore a number of important type classes in the libraries; they provide good examples of the sort of functionality which fits comfortably into a class.

27 The Functor class

In this chapter, we will introduce the important `Functor` type class.

27.1 Motivation

In Other data structures[1], we saw operations that apply to all elements of some grouped value. The prime example is `map` which works on lists. Another example we worked through was the following Tree datatype:

```
data Tree a = Leaf a | Branch (Tree a) (Tree a) deriving (Show)
```

The map function we wrote for Tree was:

```
treeMap :: (a -> b) -> Tree a -> Tree b
treeMap f (Leaf x)            = Leaf (f x)
treeMap f (Branch left right) = Branch (treeMap f left) (treeMap f right)
```

As discussed before, we can conceivably define a map-style function for any arbitrary data structure.

When we first introduced `map` in Lists II[2], we went through the process of taking a very specific function for list elements and generalizing to show how `map` combines any appropriate function with all sorts of lists. Now, we will generalize still *further*. Instead of map-for-lists and map-for-trees and other distinct maps, how about a general concept of maps for all sorts of mappable types?

27.2 Introducing `Functor`

`Functor` is a Prelude class for types which can be mapped over. It has a single method, called `fmap`. The class is defined as follows:

```
class Functor f where
    fmap :: (a -> b) -> f a -> f b
```

The usage of the type variable `f` can look a little strange at first. Here, `f` is a parametrized data type; in the signature of `fmap`, `f` takes `a` as a type parameter in one of its appearances and `b` in the other. Let's consider an instance of `Functor`: By replacing `f` with `Maybe` we get the following signature for `fmap`...

[1] Chapter 25 on page 145

[2] Chapter 13 on page 77

```
fmap :: (a -> b) -> Maybe a -> Maybe b
```

... which fits the natural definition:

```
instance Functor Maybe where
    fmap f Nothing  = Nothing
    fmap f (Just x) = Just (f x)
```

(Incidentally, this definition is in Prelude; so we didn't really need to implement `maybeMap` for that example in the "Other data structures" chapter.)

The `Functor` instance for lists (also in Prelude) is simple:

```
instance Functor [] where
    fmap = map
```

... and if we replace `f` with `[]` in the `fmap` signature, we get the familiar type of `map`.

So, `fmap` is a generalization of `map` for any parametrized data type.[3]

Naturally, we can provide `Functor` instances for our own data types. In particular, `treeMap` can be promptly relocated to an instance:

```
instance Functor Tree where
    fmap f (Leaf x)            = Leaf   (f x)
    fmap f (Branch left right) = Branch (fmap f left) (fmap f right)
```

Here's a quick demo of `fmap` in action with the instances above (to reproduce it, you only need to load the `data` and `instance` declarations for `Tree`; the others are already in Prelude):

```
• Main> fmap (2*) [1,2,3,4] [2,4,6,8]
• Main> fmap (2*) (Just 1) Just 2
• Main> fmap (fmap (2*)) [Just 1, Just 2, Just 3, Nothing] [Just 2, Just 4, Just 6, Nothing]
• Main> fmap (2*) (Branch (Branch (Leaf 1) (Leaf 2)) (Branch (Leaf 3) (Leaf 4))) Branch (Branch
  (Leaf 2) (Leaf 4)) (Branch (Leaf 6) (Leaf 8))
```

Note:
Beyond `[]` and `Maybe`, there are many other `Functor` instances already defined. Those made available from the Prelude can are listed in the Data.Functor[a] module.

[a] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Functor.html

27.2.1 The functor laws

When providing a new instance of `Functor`, you should ensure it satisfies the two functor laws. There is nothing mysterious about these laws; their role is to guarantee `fmap` behaves

[3] Data structures provide the most intuitive examples; however, there are functors which cannot reasonably be seen as data structures. A commonplace metaphor consists in thinking of functors as containers; like all metaphors, however, it can be stretched only so far.

sanely and actually performs a mapping operation (as opposed to some other nonsense). [4] The first law is:

```
fmap id = id
```

`id` is the identity function, which returns its argument unaltered. The first law states that mapping `id` over a functorial value must return the functorial value unchanged.

Next, the second law:

```
fmap (g . f) = fmap g . fmap f
```

It states that it should not matter whether we map a composed function or first map one function and then the other (assuming the application order remains the same in both cases).

27.3 What did we gain?

At this point, we can ask what benefit we get from the extra layer of generalization brought by the `Functor` class. There are two significant advantages:

- The availability of the `fmap` method relieves us from having to recall, read, and write a plethora of differently named mapping methods (`maybeMap`, `treeMap`, `weirdMap`, *ad infinitum*). As a consequence, code becomes both cleaner and easier to understand. On spotting a use of `fmap`, we instantly have a general idea of what is going on.[5] Thanks to the guarantees given by the functor laws, this general idea is surprisingly precise.
- Using the type class system, we can write `fmap`-based algorithms which work out of the box with *any* functor - be it `[]`, `Maybe`, `Tree` or whichever you need. Indeed, a number of useful classes in the core libraries inherit from `Functor`.

Type classes make it possible to create general solutions to whole categories of problems. Depending on what you use Haskell for, you may not need to define new classes often, but you will certainly be *using* type classes all the time. Many of the most powerful features and sophisticated capabilities of Haskell rely on type classes (residing either in the standard libraries or elsewhere). From this point on, classes will be a prominent presence in our studies.

[4] Some examples of nonsense that the laws rule out: removing or adding elements from a list, reversing a list, changing a `Just`-value into a `Nothing`.

[5] This is analogous to the gain in clarity provided by replacing explicit recursive algorithms on lists with implementations based on higher-order functions.

28 Monads

29 Prologue: IO, an applicative functor

For shorter links to this chapter, be them within the book or off-wiki, you can use the Haskell/Applicative prologue[1] redirect.

The emergence of functors is a watershed in the course of this book. The reasons for that will begin to reveal themselves in this prologue, as we set the stage for the next several chapters of the book. While the code examples we will work with here are very simple, we will use them to bring several new and important ideas into play, ideas that will be revisited and further developed later in the book. That being so, we recommend you to study this chapter at a gentle pace, which gives you space for thinking about the implications of each step, as well as trying out the code samples in GHCi.

29.1 Scene 1 : `Applicative`

Our initial examples will use the function `readMaybe`, which is provided by the `Text.Read` module.

```
GHCi> :m +Text.Read
GHCi> :t readMaybe
readMaybe :: Read a => String -> Maybe a
```

`readMaybe` provides a simple way of converting strings into Haskell values. If the provided string has the correct format to be read as a value of type `a`, `readMaybe` gives back the converted value wrapped in `Just`; otherwise, the result is `Nothing`.

```
GHCi> readMaybe "3" :: Maybe Integer
Just 3
GHCi> readMaybe "foo" :: Maybe Integer
Nothing
GHCi> readMaybe "3.5" :: Maybe Integer
Nothing
GHCi> readMaybe "3.5" :: Maybe Double
Just 3.5
```

Note:
To use `readMaybe`, we need to specify which type we are trying to read. Most of the time, that would be done through a combination of type inference and the signatures in our code. Occasionally, however, it is more convenient to just slap in a *type annotation* rather than writing down a proper signature. For instance, in the first example above the `:: Maybe Integer` in `readMaybe "3" :: Maybe Integer` says that the type of `readMaybe "3"` is `Maybe Integer`.

We can use `readMaybe` to write a little program in the style of those in the Simple input and output[2] chapter that:

- Gets a string given by the user through the command line;
- Tries to read it into a number (let's use `Double` as the type); and
- If the read succeeds, prints the double of the number; otherwise, prints an explanatory message and starts over.

Note:
Before continuing, we suggest you try writing the program. Beyond `readMaybe`, you will likely find `getLine`, `putStrLn` and `show` useful. Have a look at the Simple input and output[a] chapter if you need a reminder about how to do reading from and printing to the console.

[a] Chapter 10 on page 57

Here is a possible implementation:

```
import Text.Read

interactiveDoubling = do
    putStrLn "Choose a number:"
    s <- getLine
    let mx = readMaybe s :: Maybe Double
    case mx of
        Just x -> putStrLn ("The double of your number is " ++ show (2*x))
        Nothing -> do
            putStrLn "This is not a valid number. Retrying..."
            interactiveDoubling
```

```
GHCi> interactiveDoubling
Choose a number:
foo
This is not a valid number. Retrying...
Choose a number:
3
The double of your number is 6.0
```

Nice and simple. A variation of this solution might take advantage of how, given that `Maybe` is a `Functor`, we can double the value before unwrapping `mx` in the case statement:

```
interactiveDoubling = do
    putStrLn "Choose a number:"
    s <- getLine
    let mx = readMaybe s :: Maybe Double
    case fmap (2*) mx of
        Just d -> putStrLn ("The double of your number is " ++ show d)
        Nothing -> do
            putStrLn "This is not a valid number. Retrying..."
            interactiveDoubling
```

In this case, there is no real advantage in doing that. Still, keep this possibility in mind.

[2] Chapter 10 on page 57

29.1.1 Application in functors

Now, let's do something slightly more sophisticated: reading two numbers with `readMaybe` and printing their sum (we suggest that you attempt writing this one as well before continuing).

Here is one solution:

```
interactiveSumming = do
    putStrLn "Choose two numbers:"
    sx <- getLine
    sy <- getLine
    let mx = readMaybe sx :: Maybe Double
        my = readMaybe sy
    case mx of
        Just x -> case my of
            Just y -> putStrLn ("The sum of your numbers is " ++ show (x+y))
            Nothing -> retry
        Nothing -> retry
    where
    retry = do
        putStrLn "Invalid number. Retrying..."
        interactiveSumming
```

```
GHCi> interactiveSumming
Choose two numbers:
foo
4
Invalid number. Retrying...
Choose two numbers:
3
foo
Invalid number. Retrying...
Choose two numbers:
3
4
The sum of your numbers is 7.0
```

`interactiveSumming` works, but it is somewhat annoying to write. In particular, the nested `case` statements are not pretty, and make reading the code a little difficult. If only there was a way of summing the numbers before unwrapping them, analogously to what we did with `fmap` in the second version of `interactiveDoubling`, we would be able to get away with just one `case`:

```
-- Wishful thinking...
    case somehowSumMaybes mx my of
        Just z -> putStrLn ("The sum of your numbers is " ++ show (x+y))
        Nothing -> do
            putStrLn "Invalid number. Retrying..."
            interactiveSumming
```

But what should we put in place of `somehowSumMaybes`? `fmap`, for one, is not enough. While `fmap (+)` works just fine to partially apply `(+)` to the value wrapped by `Maybe`...

```
GHCi> :t (+) 3
(+) 3 :: Num a => a -> a
GHCi> :t fmap (+) (Just 3)
fmap (+) (Just 3) :: Num a => Maybe (a -> a)
```

... we don't know how to apply a function wrapped in `Maybe` to the second value. For that, we would need a function with a signature like this one...

```
(<*>) :: Maybe (a -> b) -> Maybe a -> Maybe b
```

... which would then be used like this:

```
GHCi> fmap (+) (Just 3) <*> Just 4
Just 7
```

The GHCi prompt in this example, however, is not wishful thinking: `(<*>)` actually exists, and if you try it in GHCi, it will actually work! The expression looks even neater if we use the infix synonym of `fmap`, `(<$>)`:

```
GHCi> (+) <$> Just 3 <*> Just 4
Just 7
```

The actual type `(<*>)` is more general than what we just wrote. Checking it...

```
GHCi> :t (<*>)
(<*>) :: Applicative f => f (a -> b) -> f a -> f b
```

... introduces us to a new type class: `Applicative`, the type class of *applicative functors*. For an initial explanation, we can say that an applicative functor is a functor which supports applying functions within the functor, thus allowing for smooth usage of partial application (and therefore functions of multiple arguments). All instances of `Applicative` are `Functor`s, and besides `Maybe`, there are many other common `Functor`s which are also `Applicative`.

This is the `Applicative` instance for `Maybe`:

```
instance Applicative Maybe where
    pure                  = Just
    (Just f) <*> (Just x) = Just (f x)
    _        <*> _        = Nothing
```

The definition of `(<*>)` is actually quite simple: if neither of the values are `Nothing`, apply the function `f` to `x` and wrap the result with `Just`; otherwise, give back `Nothing`. Note that the logic is exactly equivalent to what the nested `case` statement of `interactiveSumming` does.

Note that beyond `(<*>)` there is a second method in the instance above, `pure`:

```
GHCi> :t pure
pure :: Applicative f => a -> f a
```

`pure` takes a value and brings it into the functor in a default, trivial way. In the case of `Maybe`, the trivial way amounts to wrapping the value with `Just` – the nontrivial alternative would be discarding the value and giving back `Nothing`. With `pure`, we might rewrite the three-plus-four example above as...

```
GHCi> (+) <$> pure 3 <*> pure 4 :: Num a => Maybe a
Just 7
```

... or even:

```
GHCi> pure (+) <*> pure 3 <*> pure 4 :: Num a => Maybe a
Just 7
```

Just like the `Functor` class has laws which specify how sensible instance should behave, there is a set of laws for `Applicative`. Among other things, these laws specify what the ”trivial” way of bringing values into the functor through `pure` amounts to. Since there is a lot going on in this stretch of the book, we will not discuss the laws now; however, we will return to this important topic in a not too distant future.

Note:
In any case, if you are curious feel free to make a detour though the Applicative functors[a] chapter and read its ”Applicative functor laws” subsection. If you choose to go there, you might as well have a look at the ”ZipList” section, which provides an additional example of a common applicative functor that can be grasped using only what we have seen so far.

[a] Chapter 40 on page 249

To wrap things up, here is a version of `interactiveSumming` enhanced by `(<*>)`:

```
interactiveSumming = do
    putStrLn "Choose two numbers:"
    sx <- getLine
    sy <- getLine
    let mx = readMaybe sx :: Maybe Double
        my = readMaybe sy
    case (+) <$> mx <*> my of
        Just z -> putStrLn ("The sum of your numbers is " ++ show z)
        Nothing -> do
            putStrLn "Invalid number. Retrying..."
            interactiveSumming
```

29.2 Scene 2 : IO

In the examples above, we have been taking I/O actions such as `getLine` for granted. We now find ourselves at an auspicious moment to revisit a question first raised many chapters ago: what is the type of `getLine`?

Back in the Simple input and output[3] chapter, we saw the answer to that question is:

```
GHCi> :t getLine
getLine :: IO String
```

Using what we learned since then, we can now see that `IO` is a type constructor with one type variable, which happens to be instantiated as `String` in the case of `getLine`. That, however, doesn't get to the root of the issue: what does `IO String` really mean, and what is the difference between that and plain old `String`?

[3] Chapter 10 on page 57

29.2.1 Referential transparency

A key feature of Haskell is that all expressions we can write are *referentially transparent*. That means we can replace any expression whatsoever by its value without changing the behaviour of the program. For instance, consider this very simple program:

```
addExclamation :: String -> String
addExclamation s = s ++ "!"

main = putStrLn (addExclamation "Hello")
```

Its behaviour is wholly unsurprising:

```
GHCi> main
Hello!
```

Given that `addExclamation s = s ++ "!"`, we can rewrite `main` so that it doesn't mention `addExclamation`. All we have to do is replacing `s` by `"Hello"` in the right-hand side of the `addExclamation` definition and then replacing `addExclamation "Hello!"` by the resulting expression. As advertised, the program behaviour does not change:

```
GHCi> let main = putStrLn ("Hello" ++ "!")
GHCi> main
Hello!
```

Referential transparency ensures that this sort of substitution works. This guarantee extends to anywhere in any Haskell program, which goes a long way towards making programs easier to understand, and their behaviour easier to predict.

Now, suppose that the type of `getLine` were `String`. In that case, we would be able to use it as the argument to `addExclamation`, as in:

```
-- Not actual code.
main = putStrLn (addExclamation getLine)
```

In that case, however, a new question would spring forth: if `getLine` is a `String`, which `String` is it? There is no satisfactory answer: it could be `"Hello"`, `"Goodbye"`, or whatever else the user chooses to type at the terminal. And yet, *replacing* `getLine` by any `String` breaks the program, as the user would not be able to type the input string at the terminal any longer. Therefore `getLine` having type `String` would cause referential transparency to be broken. The same goes for all other I/O actions: their results are *opaque*, in that it is impossible to tell them in advance, as they depend on factors external to the program.

29.2.2 Cutting through the fog

As `getLine` illustrates, there is a fundamental indeterminacy associated with I/O actions. Respecting this indeterminacy is necessary for preserving referential transparency. In Haskell, that is achieved through the `IO` type constructor. `getLine` being an `IO String` means that it is not any actual `String`, but both a placeholder for a `String` that will only materialise when the program is executed and a promise that this `String` will indeed be delivered (in the case of `getLine`, by slurping it from the terminal). As a consequence, when we manipulate an `IO String` we are setting up plans for what *will* be done once this unknown `String` comes into being. There are quite a few ways of achieving that. In this section, we will consider two of them; to which we will add a third one in the next few chapters.

The idea of dealing a value which isn't really there might seem bizarre at first. However, we have already discussed at least one example of something not entirely unlike it without batting an eyelid. If `mx` is a `Maybe Double`, then `fmap (2*) mx` doubles the value *if it is there*, and works regardless of whether the value actually exists.[4] Both `Maybe a` and `IO a` imply, for different reasons, a layer of indirection in reaching the corresponding values of type `a`. That being so, it comes as no surprise that, like `Maybe`, *`IO` is a `Functor`*, with `fmap` being the most elementary way of getting across the indirection.

To begin with, we can exploit the fact of `IO` being a `Functor` to replace the `let` definitions in `interactiveSumming` from the end of the previous section by something more compact:

```
interactiveSumming :: IO ()
interactiveSumming = do
    putStrLn "Choose two numbers:"
    mx <- readMaybe <$> getLine -- equivalently: fmap readMaybe getLine
    my <- readMaybe <$> getLine
    case (+) <$> mx <*> my :: Maybe Double of
        Just z -> putStrLn ("The sum of your numbers is " ++ show z)
        Nothing -> do
            putStrLn "Invalid number. Retrying..."
            interactiveSumming
```

`readMaybe <$> getLine` can be read as ”once `getLine` delivers a string, whatever it turns out to be, apply `readMaybe` on it”. Referential transparency is not compromised: the value behind `readMaybe <$> getLine` is just as opaque as that of `getLine`, and its type (in this case `IO (Maybe Double)`) disallows us from replacing it with any determinate value (say, `Just 3`) that would violate referential transparency.

Beyond being a `Functor`, `IO` is also an `Applicative`, which provides us a second way of manipulating the values delivered by I/O actions. We will illustrate it with a `interactiveConcatenating` action, similar in spirit to `interactiveSumming`. A first version is just below. Can you anticipate how to simplify it with `(<*>)`?

```
interactiveConcatenating :: IO ()
interactiveConcatenating = do
    putStrLn "Choose two strings:"
    sx <- getLine
    sy <- getLine
    putStrLn "Let's concatenate them:"
    putStrLn (sx ++ sy)
```

Here is a version exploiting `(<*>)`:

```
interactiveConcatenating :: IO ()
interactiveConcatenating = do
    putStrLn "Choose two strings:"
```

[4] The key difference between the two situations is that with `Maybe` the indeterminacy is only apparent, and it is possible to figure out in advance whether there is an actual `Double` behind `mx` – or, more precisely, it is possible as long as the value of `mx` does not depend on I/O!

```
    sz <- (++) <$> getLine <*> getLine
    putStrLn "Let's concatenate them:"
    putStrLn sz
```

`(++) <$> getLine <*> getLine` is an I/O action which is made out of two other I/O actions (the two `getLine`). When it is executed, these two I/O actions are executed and the strings they deliver are concatenated. One important thing to notice is that `(<*>)` maintains a consistent order of execution between the actions it combines. Order of execution matters when dealing with I/O – examples of that are innumerable, but for starters consider this question: if we replace the second `getLine` in the example above with `take 3 <$> getLine`, which of the strings entered at the terminal will be cut down to three characters?

As `(<*>)` respects the order of actions, it provides a way of sequencing them. In particular, if we are only interested in sequencing and don't care about the result of the first action we can use `\_ y -> y` to discard it:

```
GHCi> (\_ y -> y) <$> putStrLn "First!" <*> putStrLn "Second!"
First!
Second!
```

This is such a common usage pattern that there is an operator specifically for it: `(*>)`.

```
u *> v = (\_ y -> y) <$> u <*> v
```

```
GHCi> :t (*>)
(*>) :: Applicative f => f a -> f b -> f b
GHCi> putStrLn "First!" *> putStrLn "Second!"
First!
Second!
```

It can be readily applied to `interactiveConcatenating` example:

```
interactiveConcatenating :: IO ()
interactiveConcatenating = do
    putStrLn "Choose two strings:"
    sz <- (++) <$> getLine <*> getLine
    putStrLn "Let's concatenate them:" *> putStrLn sz
```

Or, going even further:

```
interactiveConcatenating :: IO ()
interactiveConcatenating = do
    sz <- putStrLn "Choose two strings:" *> ((++) <$> getLine <*> getLine)
    putStrLn "Let's concatenate them:" *> putStrLn sz
```

Note that each of the `(*>)` replaces one of the magical line breaks of the `do` block that lead actions to be executed one after the other. In fact, that is all there is to the replaced line breaks: they are just syntactic sugar for `(*>)`.

Earlier, we said that a functor brings in a layer of indirection for accessing the values within it. The flip side of that observation is that the indirection is caused by a *context*, within which the values are found. For `IO`, the indirection is that the values are only determined when the program is executed, and the context consists in the series of instructions that will be used to produce these values (in the case of `getLine`, these instructions amount to "slurp a line of text from the terminal"). From this perspective, `(<*>)` takes two functorial values and combines not only the values within but also the contexts themselves. In the case of IO combining the contexts means appending the instructions of one I/O action to those of the other, thus sequencing the actions.

29.3 The end of the beginning

This chapter was a bit of a whirlwind! Let's recapitulate the key points we discussed in it:

- Applicative is a subclass of Functor for *applicative functors*, which are functors that support function application without leaving the functor.
- The (<*>) method of Applicative can be used as a generalisation of fmap to multiple arguments.
- An IO a is not a tangible value of type a, but a placeholder for an a value that will only come into being when the program is executed and a promise that this value will be delivered through some means. That makes referential transparency possible even when dealing with I/O actions.
- IO is a functor, and more specifically an instance of Applicative. That provides means to modify the value produced by an I/O action in spite of its indeterminacy.
- A functorial value can be seen as being made of values in a *context*. fmap cuts through the context to modify the underlying values. (<*>) combines both the contexts and the underlying values of two functorial values.
- In the case of IO, (<*>), and the closely related (*>), combine contexts by sequencing I/O actions.
- A large part of the role of do blocks is simply providing syntactic sugar for (*>).

As a final observation, note that there is still a major part of the mystery behind do blocks left to explain: what does the left arrow do? In a do-block line such as...

```
sx <- getLine
```

... it looks like we are extracting the value produced by getLine from the IO context. Thanks to the discussion about referential transparency, we now know that must be an illusion. But what is going on behind the scenes? Feel free to place your bets, as we are about to find out!

30 Understanding monads

There is a certain mystique about monads, and even about the word ”monad” itself. While one of our goals of this set of chapters is removing the shroud of mystery that is often wrapped around them, it is not difficult to understand how it comes about. Monads are very useful in Haskell, but the concept is often difficult to grasp at first. Since monads have so many applications, people often explain them from a particular point of view, which can derail your efforts towards understanding them in their full glory.

Historically, monads were introduced into Haskell to perform input and output – that is, I/O operations of the sort we dealt with in the Simple input and output[1] chapter and the prologue to this unit[2]. A predetermined execution order is crucial for things like reading and writing files, and monadic operations lend themselves naturally to sequencing. However, monads are by no means limited to input and output. They can be used to provide a whole range of features, such as exceptions, state, non-determinism, continuations, coroutines, and more. In fact, thanks to the versatility of monads, none of these constructs needed to be built into Haskell as a language; rather, they are defined by the standard libraries.

In the Prologue[3] chapter, we began with an example and used it to steadily introduce several new ideas. Here, we will do it the other way around, starting with a definition of monad and, from that, building connections with what we already know.

30.1 Definition

A *monad* is defined by three things:

- a type constructor[4] `m`;
- a function `return`;[5]
- an operator `(>>=)` which is pronounced ”bind”.

The function and operator are methods of the `Monad` type class and have types

```
return :: a -> m a
(>>=)  :: m a -> (a -> m b) -> m b
```

and are required to obey three laws[6] that will be explained later on.

[1] Chapter 10 on page 57

[2] https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue

[3] https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue

[4] Chapter 24.3 on page 142

[5] This `return` function has nothing to do with the `return` keyword found in imperative languages like C or Java; don't conflate these two.

[6] Chapter 30.3 on page 184

For a concrete example, take the `Maybe` monad. The type constructor is `m = Maybe`, while `return` and `(>>=)` are defined like this:

```
return :: a -> Maybe a
return x  = Just x

(>>=)  :: Maybe a -> (a -> Maybe b) -> Maybe b
m >>= g = case m of
            Nothing -> Nothing
            Just x  -> g x
```

`Maybe` is the monad, and `return` brings a value into it by wrapping it with `Just`. As for `(>>=)`, it takes a `m :: Maybe a` value and a `g :: a -> Maybe b` function. If `m` is `Nothing`, there is nothing to do and the result is `Nothing`. Otherwise, in the `Just x` case, `g` is applied to `x`, the underlying value wrapped in `Just`, to give a `Maybe b` result. Note that this result may or may not be `Nothing`, depending on what `g` does to `x`. To sum it all up, if there is an *underlying value* of type `a` in `m`, we apply `g` to it, which brings the underlying value back into the `Maybe` monad.

The key first step to understand how `return` and `(>>=)` work is tracking which values and arguments are monadic and which ones aren't. As in so many other cases, type signatures are our guide to the process.

30.1.1 Motivation: Maybe

To see the usefulness of `(>>=)` and the `Maybe` monad, consider the following example: Imagine a family database that provides two functions:

```
father :: Person -> Maybe Person
mother :: Person -> Maybe Person
```

These look up the name of someone's father or mother. In case our database is missing some relevant information, `Maybe` allows us to return a `Nothing` value to indicate that the lookup failed, rather than crashing the program.

Let's combine our functions to query various grandparents. For instance, the following function looks up the maternal grandfather (the father of one's mother):

```
maternalGrandfather :: Person -> Maybe Person
maternalGrandfather p =
    case mother p of
        Nothing -> Nothing
        Just mom -> father mom
```

Or consider a function that checks whether both grandfathers are in the database:

```
    bothGrandfathers :: Person -> Maybe (Person, Person)
    bothGrandfathers p =
        case father p of
            Nothing -> Nothing
            Just dad ->
                case father dad of
                    Nothing -> Nothing
                    Just gf1 ->                          -- found first
 grandfather
                        case mother p of
```

```
                           Nothing -> Nothing
                           Just mom ->
                               case father mom of
                                   Nothing -> Nothing
                                   Just gf2 ->           -- found second
grandfather
                                       Just (gf1, gf2)
```

What a mouthful! Every single query might fail by returning `Nothing` and the whole function must fail with `Nothing` if that happens.

Clearly there has to be a better way to write that instead of repeating the case of `Nothing` again and again! Indeed, that's what the `Maybe` monad is set out to do. For instance, the function retrieving the maternal grandfather has exactly the same structure as the `(>>=)` operator, so we can rewrite it as:

```
maternalGrandfather p = mother p >>= father
```

With the help of lambda expressions and `return`, we can rewrite the two grandfathers function as well:

```
bothGrandfathers p =
   father p >>=
       (\dad -> father dad >>=
           (\gf1 -> mother p >>=   -- gf1 is only used in the final return
               (\mom -> father mom >>=
                   (\gf2 -> return (gf1,gf2) ))))
```

While these nested lambda expressions may look confusing to you, the thing to take away here is that `(>>=)` releases us from listing all the `Nothing`s, shifting the focus back to the interesting part of the code.

To be a little more precise: The result of `father p` is a monadic value (in this case, either `Just dad` or `Nothing`, depending on whether `p`'s father is in the database). As the `father` function takes a regular (non-monadic value), the `(>>=)` feeds `p`'s `dad` to it *as a non-monadic* value. The result of `father dad` is then monadic again, and the process continues.

So, `(>>=)` helps us pass non-monadic values to functions without actually leaving a monad. In the case of the `Maybe` monad, the monadic aspect is the qualifier that we don't know with certainty whether the value will be found.

30.1.2 Type class

In Haskell, the `Monad` type class is used to implement monads. It is provided by the Control.Monad[7] module and included in the Prelude. The class has the following methods:

```
class Applicative m => Monad m where
    return :: a -> m a
    (>>=)  :: m a -> (a -> m b) -> m b

    (>>)   :: m a -> m b -> m b
    fail   :: String -> m a
```

[7] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Monad.html

Aside from return and bind, there are two additional methods, `(>>)` and `fail`. Both of them have default implementations, and so you don't need to provide them when writing an instance.

The operator `(>>)`, spelled "then", is a mere convenience and commonly implemented as

```
m >> n = m >>= \_ -> n
```

`(>>)` sequences two monadic actions when the second action does not involve the result of the first, which is a common scenario for monads such as `IO`.

```
printSomethingTwice :: String -> IO ()
printSomethingTwice str = putStrLn str >> putStrLn str
```

The function `fail` handles pattern match failures in `do` notation[8]. It's an unfortunate technical necessity and doesn't really have anything to do with monads. You are advised not to call `fail` directly in your code.

30.1.3 `Monad` and `Applicative`

An important thing to note is that `Applicative` is a superclass of `Monad`.[9] That has a few consequences worth highlighting. First of all, every `Monad` is also a `Functor` and an `Applicative`, and so `fmap`, `pure`, `(<*>)` can all be used with monads. Secondly, actually writing a `Monad` instance also requires providing `Functor` and `Applicative` instances. We will discuss ways of doing that later in this chapter. Thirdly, if you have worked through the Prologue[10], the types and roles of `return` and `(>>)` should look familiar...

```
(*>) :: Applicative f => f a -> f b -> f b
(>>) :: Monad m => m a -> m b -> m b

pure :: Applicative f => a -> f a
return :: Monad m => a -> m a
```

The only difference between the types of `(*>)` and `(>>)` is that the constraint changes from `Applicative` to `Monad`. In fact, that is the only difference between the methods: if you are dealing with a `Monad` you can always replace `(*>)` and `(>>)`, and vice-versa. The same goes for `pure`/`return` – in fact, it is not even necessary to implement `return` if there is an independent definition of `pure` in the `Applicative` instance, as `return = pure` is provided as a default definition of `return`.

30.2 Notions of Computation

We have seen how `(>>=)` and `return` are very handy for removing boilerplate code that crops up when using `Maybe`. That, however, is not enough to justify why monads matter so

[8] Chapter 33 on page 197

[9] This important superclass relationship was, thanks to historic accidents, only implemented quite recently (early 2015) in GHC (version 7.10). If you are using a GHC version older than that, this class constraint will not exist, and so some of the practical considerations we will make next will not apply.

[10] https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue

much. Our next step towards that will be rewriting the two-grandfathers function in a quite different-looking style: using `do` notation with explicit braces and semicolons[11]. Depending on your experience with other programming languages, you may find this very suggestive:

```
bothGrandfathers p = do {
    dad <- father p;
    gf1 <- father dad;
    mom <- mother p;
    gf2 <- father mom;
    return (gf1, gf2);
  }
```

If this looks like a code snippet of an imperative programming language to you, that's because it is. In particular, this imperative language supports *exceptions* : `father` and `mother` are functions that might fail to produce results, i.e. raise an exception, and when that happens, the whole `do`-block will fail, i.e. terminate with an exception.

In other words, the expression `father p`, which has type `Maybe Person`, is interpreted as a statement of an imperative language that returns a `Person` as result. This is true for all monads: a value of type `M a` is interpreted as a statement of an imperative language that returns a value of type `a` as result; and the semantics of this language are determined by the monad `M`.[12]

Under this interpretation, the bind operator `(>>=)` is simply a function version of the semicolon. Just like a `let` expression can be written as a function application,

```
    let x = foo in x + 3          corresponds to      (\x -> x + 3) foo
```

an assignment and semicolon can be written as the bind operator:

```
    x <- foo; return (x + 3)      corresponds to      foo >>= (\x -> return (x +
  3))
```

The `return` function lifts a plain value `a` to `M a`, a statement of the imperative language corresponding to the monad `M`.

Note:
The fact that `(>>=)`, and therefore `Monad`, lies behind the left arrows in `do`-blocks explains why we were not able to explain them in the Prologue[a], when we only knew about `Functor` and `Applicative`. `Applicative` would be enough to provide some, but not all, of the functionality of a `do`-block.

[a] https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue%23The%20end%20of%20the%20beginning

Different semantics of the imperative language correspond to different monads. The following table shows the classic selection that every Haskell programmer should know. If the

[11] Chapter 23.2 on page 134

[12] By "semantics", we mean what the language allows you to say. In the case of `Maybe`, the semantics allow us to express failure, as statements may fail to produce a result, leading to the statements that follow it being skipped.

idea behind monads is still unclear to you, studying each of the examples in the following chapters will not only give you a well-rounded toolbox but also help you understand the common abstraction behind them.

Monad	Imperative Semantics	Wikibook chapter
`Maybe`	Exception (anonymous)	Haskell/Understanding monads/Maybe[13]
`Error`	Exception (with error description)	Haskell/Understanding monads/Error[14]
`State`	Global state	Haskell/Understanding monads/State[15]
`IO`	Input/Output	Haskell/Understanding monads/IO[16]
`[]` (lists)	Nondeterminism	Haskell/Understanding monads/List[17]
`Reader`	Environment	Haskell/Understanding monads/Reader[18]
`Writer`	Logger	Haskell/Understanding monads/Writer[19]

Furthermore, these different semantics need not occur in isolation. As we will see in a few chapters, it is possible to mix and match them by using monad transformers[20] to combine the semantics of multiple monads in a single monad.

30.3 Monad Laws

In Haskell, every instance of the `Monad` type class (and thus all implementations of bind `(>>=)` and `return`) must obey the following three laws:

```
m >>= return     =  m                        -- right unit
return x >>= f   =  f x                      -- left unit

(m >>= f) >>= g  =  m >>= (\x -> f x >>= g)  -- associativity
```

30.3.1 Return as neutral element

The behavior of `return` is specified by the left and right unit laws. They state that `return` doesn't perform any computation, it just collects values. For instance,

```
maternalGrandfather p = do
        mom <- mother p
        gf  <- father mom
        return gf
```

[13] Chapter 31 on page 189
[14] https://en.wikibooks.org/wiki/Haskell%2FUnderstanding%20monads%2FError
[15] Chapter 35 on page 207
[16] Chapter 34 on page 199
[17] Chapter 32 on page 193
[18] https://en.wikibooks.org/wiki/Haskell%2FUnderstanding%20monads%2FReader
[19] https://en.wikibooks.org/wiki/Haskell%2FUnderstanding%20monads%2FWriter
[20] Chapter 37 on page 229

is exactly the same as

```
maternalGrandfather p = do
        mom  <- mother p
        father mom
```

by virtue of the right unit law.

30.3.2 Associativity of bind

The law of associativity makes sure that (like the semicolon) the bind operator `(>>=)` only cares about the order of computations, not about their nesting; e.g. we could have written `bothGrandfathers` like this (compare with our earliest version without `do`):

```
bothGrandfathers p =
   (father p >>= father) >>=
       (\gf1 -> (mother p >>= father) >>=
           (\gf2 -> return (gf1,gf2) ))
```

The associativity of the *then* operator `(>>)` is a special case:

```
(m >> n) >> o  =  m >> (n >> o)
```

Monadic composition

It is easier to picture the associativity of bind by recasting the law as

```
(f >=> g) >=> h  =  f >=> (g >=> h)
```

where `(>=>)` is the *monad composition operator*, a close analogue of the function composition operator `(.)`, only with flipped arguments. It is defined as:

```
(>=>) :: Monad m => (a -> m b) -> (b -> m c) -> a -> m c
f >=> g = \x -> f x >>= g
```

There is also `(<=<)`, which is flipped version of `(>=>)`. When using it, the order of composition matches that of `(.)`, so that in `(f <=< g)` `g` comes first.[21]

30.4 Monads and Category Theory

Monads originally come from a branch of mathematics called Category Theory. Fortunately, it is entirely unnecessary to understand category theory in order to understand and use monads in Haskell. The definition of monads in Category Theory actually uses a slightly different presentation. Translated into Haskell, this presentation gives an alternative yet equivalent definition of a monad which can give us some additional insight.[22]

[21] Of course, the functions in regular function composition are non-monadic functions whereas monadic composition takes only monadic functions.

[22] Deep into the Advanced Track, we will cover the theoretical side of the story in the chapter on Category Theory ^{Chapter59.3 on page 439}.

So far, we have defined monads in terms of `(>>=)` and `return`. The alternative definition, instead, treats monads as functors with two additional combinators:

```
fmap   :: (a -> b) -> M a -> M b  -- functor

return :: a -> M a
join   :: M (M a) -> M a
```

For the purposes of this discussion, we will use the functors-as-containers metaphor discussed in the chapter on the functor class[23]. According to it, a functor `M` can be thought of as container, so that `M a` ”contains” values of type `a`, with a corresponding mapping function, i.e. `fmap`, that allows functions to be applied to values inside it.

Under this interpretation, the functions behave as follows:

- `fmap` applies a given function to every element in a container
- `return` packages an element into a container,
- `join` takes a container of containers and flattens it into a single container.

With these functions, the bind combinator can be defined as follows:

```
m >>= g = join (fmap g m)
```

Likewise, we could give a definition of `fmap` and `join` in terms of `(>>=)` and `return`:

```
fmap f x = x >>= (return . f)
join x   = x >>= id
```

30.5 `liftM` and Friends

Earlier, we pointed out that every `Monad` is an `Applicative`, and therefore also a `Functor`. One of the consequences of that was `return` and `(>>)` being monad-only versions of `pure` and `(*>)` respectively. It doesn't stop there, though. For one, `Control.Monad` defines `liftM`, a function with a strangely familiar type signature...

```
liftM :: (Monad m) => (a1 -> r) -> m a1 -> m r
```

As you might suspect, `liftM` is merely `fmap` implemented with `(>>=)` and `return`, just as we have done in the previous section. `liftM` and `fmap` are therefore interchangeable.

Another `Control.Monad` function with an uncanny type is `ap`:

```
ap :: Monad m => m (a -> b) -> m a -> m b
```

Analogously to the other cases, `ap` is a monad-only version of `(<*>)`.

There are quite a few more examples of `Applicative` functions that have versions *specialised* to `Monad` in `Control.Monad` and other base library modules. Their existence is primarily due to historical reasons: several years went by between the introductions of

[23] Chapter 27 on page 163

Monad and Applicative in Haskell, and it took an even longer time for Applicative to become a superclass of Monad, thus making usage of the specialised variants optional. While in principle there is little need for using the monad-only versions nowadays, in practice you will see return and (>>) all the time in other people's code – at this point, their usage is well established thanks to more than two decades of Haskell praxis without Applicative being a superclass of Monad.

Note:
Given that Applicative is a superclass of Monad, the most obvious way of implementing Monad begins by writing the Functor instance and then moving down the class hierarchy:

```
instance Functor Foo where
    fmap = -- etc.

instance Applicative Foo where
    pure = -- etc.
    (<*>) = -- etc.

instance Monad Foo where
    (>>=) = -- etc.
```

While following the next few chapters, you will likely want to write instances of Monad and try them out, be it to run the examples in the book or to do other experiments you might think of. However, writing the instances in the manner shown above requires implementing pure and (<*>), which is not a comfortable task at this point of the book as we haven't covered the Applicative laws yet (we will only do so at the applicative functors chapter[a]). Fortunately, there is a workaround: implementing just (>>=) and return, thus providing a self-sufficient Monad instance, and then using liftM, ap and return to fill in the other instances:

```
instance Monad Foo where
    return = -- etc.
    (>>=) = -- etc.

instance Applicative Foo where
    pure = return
    (<*>) = ap

instance Functor Foo where
    fmap = liftM
```

The examples and exercises in this initial series of chapters about monads will not demand writing Applicative instances, and so you can use this workaround until we discuss Applicative in detail.

[a] Chapter 40 on page 249

31 The Maybe monad

We introduced monads using `Maybe` as an example. The `Maybe` monad represents computations which might ”go wrong” by not returning a value. For reference, here are the definitions of `return` and `(>>=)` for `Maybe` as we saw in the last chapter:[1]

```
return :: a -> Maybe a
return x  = Just x

(>>=)  :: Maybe a -> (a -> Maybe b) -> Maybe b
(>>=) m g = case m of
               Nothing -> Nothing
               Just x  -> g x
```

31.1 Safe functions

The `Maybe` datatype provides a way to make a safety wrapper around *partial functions*, that is, functions which can fail to work for a range of arguments. For example, `head` and `tail` only work with non-empty lists. Another typical case, which we will explore in this section, are mathematical functions like `sqrt` and `log`; (as far as real numbers are concerned) these are only defined for non-negative arguments.

```
> log 1000
6.907755278982137
> log (-1000)
''ERROR'' -- runtime error
```

To avoid this crash, a ”safe” implementation of log could be:

```
safeLog :: (Floating a, Ord a) => a -> Maybe a
safeLog x
    | x > 0     = Just (log x)
    | otherwise = Nothing
```

```
> safeLog 1000
Just 6.907755278982137
> safeLog -1000
Nothing
```

We could write similar ”safe functions” for all functions with limited domains such as division, square-root, and inverse trigonometric functions (`safeDiv`, `safeSqrt`, `safeArcSin`, etc. all of which would have the same *type* as `safeLog` but definitions specific to their constraints)

[1] The definitions in the actual instance in `Data.Maybe` are written a little differently, but are fully equivalent to these.

If we wanted to combine these monadic functions, the cleanest approach is with monadic composition (which was mentioned briefly near the end of the last chapter[2]) and point-free style:

```
safeLogSqrt = safeLog <=< safeSqrt
```

Written in this way, `safeLogSqrt` resembles a lot its unsafe, non-monadic counterpart:

```
unsafeLogSqrt = log . sqrt
```

31.2 Lookup tables

A lookup table relates *keys* to *values*. You *look up* a value by knowing its key and using the lookup table. For example, you might have a phone book application with a lookup table where contact names are keys to corresponding phone numbers. An elementary way of implementing lookup tables in Haskell is to use a list of pairs: `[(a, b)]`. Here `a` is the type of the keys, and `b` the type of the values.[3] Here's how the phone book lookup table might look:

```
phonebook :: [(String, String)]
phonebook = [ ("Bob",   "01788 665242"),
              ("Fred",  "01624 556442"),
              ("Alice", "01889 985333"),
              ("Jane",  "01732 187565") ]
```

The most common thing you might do with a lookup table is look up values. Everything is fine if we try to look up ”Bob”, ”Fred”, ”Alice” or ”Jane” in our phone book, but what if we were to look up ”Zoe”? Zoe isn't in our phone book, so the lookup would fail. Hence, the Haskell function to look up a value from the table is a `Maybe` computation (it is available from Prelude):

```
lookup :: Eq a => a  -- a key
       -> [(a, b)]   -- the lookup table to use
       -> Maybe b    -- the result of the lookup
```

Let us explore some of the results from lookup:

```
Prelude> lookup "Bob" phonebook
Just "01788 665242"
Prelude> lookup "Jane" phonebook
Just "01732 187565"
Prelude> lookup "Zoe" phonebook
Nothing
```

Now let's expand this into using the full power of the monadic interface. Say, we're now working for the government, and once we have a phone number from our contact, we want to look up this phone number in a big, government-sized lookup table to find out the

[2] Chapter 30.3.2 on page 185

[3] Check the chapter about maps ˆ{Chapter74 on page 503} in Haskell in Practice for a different, and potentially more useful, implementation.

registration number of their car. This, of course, will be another Maybe-computation. But if the person we're looking for isn't in our phone book, we certainly won't be able to look up their registration number in the governmental database. What we need is a function that will take the results from the first computation and put it into the second lookup *only* if we get a successful value in the first lookup. Of course, our final result should be Nothing if we get Nothing from either of the lookups.

```
getRegistrationNumber :: String       -- their name
                      -> Maybe String -- their registration number
getRegistrationNumber name =
  lookup name phonebook >>=
    (\number -> lookup number governmentDatabase)
```

If we then wanted to use the result from the governmental database lookup in a third lookup (say we want to look up their registration number to see if they owe any car tax), then we could extend our getRegistrationNumber function:

```
getTaxOwed :: String       -- their name
           -> Maybe Double -- the amount of tax they owe
getTaxOwed name =
  lookup name phonebook >>=
    (\number -> lookup number governmentDatabase) >>=
      (\registration -> lookup registration taxDatabase)
```

Or, using the do-block style:

```
getTaxOwed name = do
  number       <- lookup name phonebook
  registration <- lookup number governmentDatabase
  lookup registration taxDatabase
```

Let's just pause here and think about what would happen if we got a Nothing anywhere. By definition, when the first argument to >>= is Nothing, it just returns Nothing while ignoring whatever function it is given. Thus, a Nothing at *any stage* in the large computation will result in a Nothing overall, regardless of the other functions. After the first Nothing hits, all >>=s will just pass it to each other, skipping the other function arguments. The technical description says that the structure of the Maybe monad *propagates failures.*

31.3 Open monads

Another trait of the Maybe monad is that it is "open": if we have a Just value, we can see the contents and extract the associated values through pattern matching.

```
zeroAsDefault :: Maybe Int -> Int
zeroAsDefault mx = case mx of
    Nothing -> 0
    Just x -> x
```

This usage pattern of replacing Nothing with a default is captured by the fromMaybe function in Data.Maybe.

```
zeroAsDefault :: Maybe Int -> Int
zeroAsDefault mx = fromMaybe 0 mx
```

The maybe Prelude function allows us to do it in a more general way, by supplying a function to modify the extracted value.

```
displayResult :: Maybe Int -> String
displayResult mx = maybe "There was no result" (("The result was " ++) . show)
 mx
```

```
Prelude> :t maybe
maybe :: b -> (a -> b) -> Maybe a -> b
Prelude> displayResult (Just 10)
"The result was 10"
Prelude> displayResult Nothing
"There was no result"
```

This possibility makes sense for Maybe, as it allows us to recover from failures. Not all monads are open in this way; often, they are designed to hide unnecessary details. return and (>>=) alone do not allow us to extract the underlying value from a monadic computation, and so it is perfectly possible to make a ”no-exit” monad, from which it is never possible to extract values. The most obvious example of that is the IO monad.

31.4 Maybe and safety

We have seen how Maybe can make code safer by providing a graceful way to deal with failure that does not involve runtime errors. Does that mean we should always use Maybe for *everything*? Not really.

When you write a function, you are able to tell whether it might fail to produce a result during normal operation of the program,[4] either because the functions you use might fail (as in the examples in this chapter) or because you know some of the argument or intermediate result values do not make sense (for instance, imagine a calculation that is only meaningful if its argument is less than 10). If that is the case, by all means use Maybe to signal failure; it is far better than returning an arbitrary default value or throwing an error.

Now, adding Maybe to a result type without a reason would only make the code more confusing and no safer. The type signature of a function with unnecessary Maybe would tell users of the code that the function could fail when it actually can’t. Of course, that is not as bad a lie as the opposite one (that is, claiming that a function will not fail when it actually can), but we really want honest code in *all* cases. Furthermore, using Maybe forces us to propagate failure (with fmap or monadic code) and eventually handle the failure cases (using pattern matching, the maybe function, or fromMaybe from Data.Maybe). If the function cannot actually fail, coding for failure is an unnecessary complication.

[4] With ”normal operation” we mean to exclude failure caused by uncontrollable circumstances in the real world, such as memory exhaustion or a dog chewing the printer cable.

32 The List monad

Lists are a fundamental part of Haskell, and we've used them extensively before getting to this chapter. The novel insight is that the list type is a monad too!

As monads, lists are used to model *nondeterministic* computations which may return an arbitrary number of results. There is a certain parallel with how `Maybe` represented computations which could return zero or one value; but with lists, we can return zero, one, or many values (the number of values being reflected in the length of the list).

32.1 The `Monad` instance of lists

The `return` function for lists simply injects a value into a list:

```
return x = [x]
```

In other words, `return` here makes a list containing one element, namely the single argument it took. The type of the *list return* is `return :: a -> [a]`, or, equivalently, `return :: a -> [] a`. The latter style of writing it makes it more obvious that we are replacing the generic type constructor in the signature of `return` (which we had called `M` in Understanding monads[1]) by the list type constructor `[]` (which is distinct from but easy to confuse with the empty list!).

The binding operator is less trivial. We will begin by considering its type, which for the case of lists should be:

```
[a] -> (a -> [b]) -> [b]
```

This is just what we'd expect: it pulls out the values from the list to give them to a function that produces a new list.

The actual process here involves first `map`ping a given function over a given list to get back a list of lists, i.e. type `[[b]]` (of course, many functions which you might use in mapping do not return lists; but, as shown in the type signature above, monadic binding for lists only works with functions that return lists). To get back to a regular list, we then concatenate the elements of our list of lists to get a final result of type `[b]`. Thus, we can define the list version of `(>>=)`:

```
xs >>= f = concat (map f xs)
```

The bind operator is key to understanding how different monads do their jobs, as its definition specifies the chaining strategy used when working with the monad. In the case of

[1] Chapter 30 on page 179

the list monad, the strategy allows us to model non-determinism: an `a -> [b]` function can be seen as a way of generating, from an input of type `a`, an unspecified number of *possible* outputs of type `b`, without settling on any one of them in particular. `(>>=)`, from that perspective, does that for multiple inputs and combines all output possibilities in a single result list.

32.2 Bunny invasion

It is easy to incorporate the familiar list processing functions in monadic code. Consider this example: rabbits raise an average of six kits in each litter, half of which will be female. Starting with a single mother, we can model the number of female kits in each successive generation (i.e. the number of new kits after the rabbits grow up and have their own litters):

```
Prelude> let generation = replicate 3
Prelude> ["bunny"] >>= generation
["bunny","bunny","bunny"]
Prelude> ["bunny"] >>= generation >>= generation
["bunny","bunny","bunny","bunny","bunny","bunny","bunny","bunny","bunny"]
```

In this silly example all elements are equal, but the same overall logic could be used to model radioactive decay[2], or chemical reactions, or any phenomena that produces a series of elements starting from a single one.

Exercises:

1. Predict what should be the result of `["bunny", "rabbit"] >>= generation`.
2. Implement `themselvesTimes :: [Int] -> [Int]`, which takes each number n in the argument list and generates n copies of it in the result list.

32.3 Board game example

Suppose we are modeling a turn-based board game and want to find all the possible ways the game could progress. We would need a function to calculate the list of options for the next turn, given a current board state:

```
nextConfigs :: Board -> [Board]
nextConfigs bd = undefined -- details not important
```

To figure out all the possibilities after two turns, we would again apply our function to each of the elements of our new list of board states. Our function takes a single board state and returns a list of possible new states. Thus, we can use monadic binding to map the function over each element from the list:

```
nextConfigs bd >>= nextConfigs
```

[2] http://en.wikipedia.org/wiki/Decay_chain

In the same fashion, we could bind the result back to the function yet again (ad infinitum) to generate the next turn's possibilities. Depending on the particular game's rules, we may reach board states that have no possible next-turns; in those cases, our function will return the empty list.

On a side note, we could translate several turns into a do block (like we did for the grandparents example in Understanding monads[3]):

```
threeTurns :: Board -> [Board]
threeTurns bd = do
  bd1 <- nextConfigs bd  -- bd1 refers to a board configuration after 1 turn
  bd2 <- nextConfigs bd1
  nextConfigs bd2
```

If the above looks too magical, keep in mind that do notation is syntactic sugar for (>>=) operations. To the right of each left-arrow, there is a function with arguments that evaluate to a list; the variable to the left of the arrow stands for the list elements. After a left-arrow assignment line, there can be later lines that call the assigned variable as an argument for a function. This later function will be performed for *each* of the elements from within the list that came from the left-arrow line's function. This per-element process corresponds to the ‘map‘ in the definition of (>>=). A resulting list of lists (one per element of the original list) will be flattened into a single list (the ‘concat‘ in the definition of (>>=)).

32.4 List comprehensions

The list monad works in a way that has uncanny similarity to list comprehensions. Let's slightly modify the do block we just wrote for threeTurns so that it ends with a return...

```
threeTurns bd = do
  bd1 <- nextConfigs bd
  bd2 <- nextConfigs bd1
  bd3 <- nextConfigs bd2
  return bd3
```

This mirrors exactly the following list comprehension:

```
threeTurns bd = [ bd3 | bd1 <- nextConfigs bd, bd2 <- nextConfigs bd1, bd3 <-
 nextConfigs bd2 ]
```

(In a list comprehension, it is perfectly legal to use the elements drawn from one list to define the following ones, like we did here.)

The resemblance is no coincidence: list comprehensions are, behind the scenes, defined in terms of concatMap, a function available from the Prelude that is defined as concatMap f xs = concat (map f xs). That's just the list monad binding definition again! To summarize the nature of the list monad: binding for the list monad *is* a combination of concatenation and mapping, and so the combined function concatMap *is* effectively the same as >>= for lists (except for different syntactic order).

[3] Chapter 30 on page 179

For the correspondence between list monad and list comprehension to be complete, we need a way to reproduce the filtering that list comprehensions can do. We will explain how that can be achieved a little later in the Additive monads[4] chapter.

Exercises:
As discussed in Understanding monads[a], all `Monad`s also have an instance of `Applicative`. In particular, `(<*>)` for that instance might be defined as:

```
fs <*> xs = concatMap (\f -> map f xs) fs
```

1. Explain briefly what this `(<*>)` does.
2. Write an alternative definition of `(<*>)` using a list comprehension. Do not use `map`, `concat` or `concatMap` explicitly.

[a] Chapter 30 on page 179

[4] https://en.wikibooks.org/wiki/Haskell%2FMonadPlus

33 `do` Notation

1. REDIRECT Haskell/do notation[1]

[1] Chapter 33 on page 197

34 The IO monad

Two defining features of Haskell are *pure functions* and *lazy evaluation*. All Haskell functions are pure, which means that, when given the same arguments, they return the same results. Lazy evaluation means that, by default, Haskell values are only evaluated when some part of the program requires them – perhaps never, if they are never used – and repeated evaluation of the same value is avoided wherever possible.

Pure functions and lazy evaluation bring forth a number of advantages. In particular, pure functions are reliable and predictable; they ease debugging and validation. Test cases can also be set up easily since we can be sure that nothing other than the arguments will influence a function's result. Being entirely contained within the program, the Haskell compiler can evaluate functions thoroughly in order to optimize the compiled code. However, input and output operations, which involve interaction with the world outside the confines of the program, can't be expressed through pure functions. Furthermore, in most cases I/O can't be done lazily. Since lazy computations are only performed when their values become necessary, unfettered lazy I/O would make the order of execution of the real world effects unpredictable.

There is no way to ignore this issue, as any useful program needs to do I/O, even if it is only to display a result. That being so, how do we manage actions like opening a network connection, writing a file, reading input from the outside world, or anything else that goes beyond calculating a value? The main insight is: *actions are not functions*. The `IO` type constructor provides a way to represent actions as Haskell values, so that we can manipulate them with pure functions. In the Prologue[1] chapter, we anticipated some of the key features of this solution. Now that we also know that `IO` is a monad, we can wrap up the discussion we started there.

34.1 Combining functions and I/O actions

Let's combine functions with I/O to create a full program that will:

1. Ask the user to insert a string
2. Read their string
3. Use `fmap` to apply a function `shout` that capitalizes all the letters from the string
4. Write the resulting string

```
module Main where

import Data.Char (toUpper)
import Control.Monad
```

[1] https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue

```
main = putStrLn "Write your string: " >> fmap shout getLine >>= putStrLn

shout = map toUpper
```

We have a full-blown program, but we didn't include any type definitions. Which parts are functions and which are IO actions or other values? We can load our program in GHCi and check the types:

```
main :: IO ()
putStrLn :: String -> IO ()
"Write your string: " :: [Char]
(>>) :: Monad m => m a -> m b -> m b
fmap :: Functor m => (a -> b) -> m a -> m b
shout :: [Char] -> [Char]
getLine :: IO String
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

Whew, that is a lot of information there. We've seen all of this before, but let's review.

`main` is `IO ()`. That's not a function. Functions are of types `a -> b`. Our entire program is an IO action.

`putStrLn` is a function, but it results in an IO action. The "Write your string: " text is a `String` (remember, that's just a synonym for `[Char]`). It is used as an argument for `putStrLn` and is incorporated into the IO action that results. So, `putStrLn` is a function, but `putStrLn x` evaluates to an IO action. The `()` part of the IO type indicates that nothing is available to be passed on to any later functions or actions.

That last part is key. We sometimes say informally that an IO action "returns" something; however, taking that too literally leads to confusion. It is clear what we mean when we talk about *functions* returning results, but IO actions are not functions. Let's skip down to `getLine` — an IO action that *does* provide a value. `getLine` is not a function that returns a `String` because *`getLine` isn't a function.* Rather, `getLine` is an IO action which, when evaluated, will materialize a `String`, which can then be passed to later functions through, for instance, `fmap` and `(>>=)`.

When we use `getLine` to get a `String`, the value is monadic because it is wrapped in `IO` functor (which happens to be a monad). We cannot pass the value directly to a function that takes plain (non-monadic, or non-functorial) values. `fmap` does the work of taking a non-monadic function while passing in and returning monadic values.

As we've seen already, `(>>=)` does the work of passing a monadic value into a function that takes a non-monadic value and returns a monadic value. It may seem inefficient for `fmap` to take the non-monadic result of its given function and return a monadic value only for `(>>=)` to then pass the underlying non-monadic value to the next function. It is precisely this sort of chaining, however, that creates the reliable sequencing that make monads so effective at integrating pure functions with IO actions.

34.1.1 *do* notation review

Given the emphasis on sequencing, the `do` notation[2] can be especially appealing with the `IO` monad. Our program

```
putStrLn "Write your string: " >> fmap shout getLine >>= putStrLn
```

could be written as:

```
do putStrLn "Write your string: "
   string <- getLine
   putStrLn (shout string)
```

34.2 The universe as part of our program

One way of viewing the `IO` monad is to consider `IO a` as a computation which provides a value of type `a` while changing *the state of the world* by doing input and output. Obviously, you cannot literally set the state of the world; it is hidden from you, as the `IO` functor is abstract (that is, you cannot dig into it to see the underlying values; it is closed in a way opposite to that in which `Maybe` can be said to be open[3]).

Understand that this idea of the universe as an object affected and affecting Haskell values through `IO` is only a metaphor; a loose interpretation at best. The more mundane fact is that `IO` simply brings some very base-level operations into the Haskell language.[4] Remember that Haskell is an abstraction, and that Haskell programs must be compiled to machine code in order to actually run. The actual workings of IO happen at a lower level of abstraction, and are wired into the very definition of the Haskell language.[5]

34.3 Pure and impure

The adjectives ”pure” and ”impure” often crop up while talking about I/O in Haskell. To clarify what is meant by them, we will revisit the discussion about referential transparency from the Prologue chapter[6]. Consider the following snippet:

```
speakTo :: (String -> String) -> IO String
speakTo fSentence = fmap fSentence getLine

-- Usage example.
sayHello :: IO String
sayHello = speakTo (\name -> "Hello, " ++ name ++ "!")
```

[2] Chapter 33 on page 197

[3] Chapter 31.3 on page 191

[4] The technical term is ”primitive”, as in primitive operations.

[5] The same can be said about all higher-level programming languages, of course. Incidentally, Haskell's IO operations can actually be extended via the *Foreign Function Interface* (FFI) which can make calls to C libraries. As C can use inline assembly code, Haskell can indirectly engage with anything a computer can do. Still, Haskell functions manipulate such outside operations only *indirectly* as values in `IO` functors.

[6] https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue%23Referential%20transparency

In most other programming languages, which do not have separate types for I/O actions, `speakTo` would have a type akin to:

```
speakTo :: (String -> String) -> String
```

With such a type, however, `speakTo` would not be a function at all! Functions produce the same results when given the same arguments; the `String` delivered by `speakTo`, however, also depends on whatever is typed at the terminal prompt. In Haskell, we avoid that pitfall by returning an `IO String`, which is not a `String` but a promise that *some* `String` will be delivered by carrying out certain instructions involving I/O (in this case, the I/O consists of getting a line of input from the terminal). Though the `String` can be different each time `speakTo` is evaluated, the I/O instructions are always the same.

When we say Haskell is a purely functional language, we mean that all of its functions are *really* functions – or, in other words, that Haskell expressions are always referentially transparent. If `speakTo` had the problematic type we mentioned above, referential transparency would be violated: `sayHello` would be a `String`, and yet replacing it by any specific string would break the program.

In spite of Haskell being purely functional, `IO` actions can be said to be *impure* because their impact on the outside world are *side effects* (as opposed to the regular effects that are entirely contained within Haskell). Programming languages that lack purity may have side-effects in many other places connected with various calculations. Purely functional languages, however, assure that *even expressions with impure values are referentially transparent.* That means we can talk about, reason about and handle impurity in a purely functional way, using purely functional machinery such as functors and monads. While `IO` actions are impure, all of the Haskell functions that manipulate them remain pure.

Functional purity, coupled to the fact that I/O shows up in types, benefit Haskell programmers in various ways. The guarantees about referential transparency increase a lot the potential for compiler optimizations. `IO` values being distinguishable through types alone make it possible to immediately tell where we are engaging with side effects or opaque values. As `IO` itself is just another functor, we maintain to the fullest extent the predictability and ease of reasoning associated with pure functions.

34.4 Functional and imperative

When we introduced monads[7], we said that a monadic expression can be interpreted as a statement of an imperative language. That interpretation is immediately compelling for `IO`, as the language around IO actions looks a lot like a conventional imperative language. It must be clear, however, that we are talking about an *interpretation*. We are not saying that monads or `do` notation turn Haskell into an imperative language. The point is merely that you can view and understand monadic code in terms of imperative statements. The semantics may be imperative, but the implementation of monads and `(>>=)` is still purely functional. To make this distinction clear, let's look at a little illustration:

[7] Chapter 30.2 on page 182

```
int x;
scanf("%d", &x);
printf("%d\n", x);
```

This is a snippet of C, a typical imperative language. In it, we declare a variable `x`, read its value from user input with `scanf` and then print it with `printf`. We can, within an `IO` do block, write a Haskell snippet that performs the same function and looks quite similar:

```
x <- readLn
print x
```

Semantically, the snippets are nearly equivalent.[8] In the C code, however, the statements directly correspond to instructions to be carried out by the program. The Haskell snippet, on the other hand, is desugared to:

```
readLn >>= \x -> print x
```

The desugared version has no statements, only functions being applied. We tell the program the order of the operations indirectly as a simple consequence of *data dependencies*: when we chain monadic computations with `(>>=)`, we get the later results by applying functions to the results of the earlier ones. It just happens that, for instance, evaluating `print x` leads to a string to be printed in the terminal.

When using monads, Haskell allows us to write code with imperative semantics while keeping the advantages of functional programming.

34.5 I/O in the libraries

So far the only I/O primitives we have used were `putStrLn` and `getLine` and small variations thereof. The standard libraries, however, offer many other useful functions and actions involving `IO`. We present some of the most important ones in the IO chapter in Haskell in Practice[9], including the basic functionality needed for reading from and writing to files.

34.6 Monadic control structures

Given that monads allow us to express sequential execution of actions in a wholly general way, could we use them to implement common iterative patterns, such as loops? In this section, we will present a few of the functions from the standard libraries which allow us to do precisely that. While the examples are presented here applied to `IO`, keep in mind that the following ideas apply to *every* monad.

[8] One difference is that `x` is a mutable variable in C, and so it is possible to declare it in one statement and set its value in the next; Haskell never allows such mutability. If we wanted to imitate the C code even more closely, we could have used an `IORef`, which is a cell that contains a value which can be destructively updated. For obvious reasons, `IORef`s can only be used within the `IO` monad.

[9] Chapter 75 on page 505

Remember, there is nothing magical about monadic values; we can manipulate them just like any other values in Haskell. Knowing that, we might think to try the following function to get five lines of user input:

```
fiveGetLines = replicate 5 getLine
```

That won't do, however (try it in GHCi!). The problem is that `replicate` produces, in this case, a list of actions, while we want an action which returns a list (that is, `IO [String]` rather than `[IO String]`). What we need is a *fold* to run down the list of actions, executing them and combining the results into a single list. As it happens, there is a Prelude function which does that: `sequence`.

```
sequence :: (Monad m) => [m a] -> m [a]
```

And so, we get the desired action with:

```
fiveGetLines = sequence $ replicate 5 getLine
```

`replicate` and `sequence` form an appealing combination; so Control.Monad[10] offers a `replicateM` function for repeating an action an arbitrary number of times. `Control.Monad` provides a number of other convenience functions in the same spirit - monadic zips, folds, and so forth.

```
fiveGetLinesAlt = replicateM 5 getLine
```

A particularly important combination is `map` and `sequence`. Together, they allow us to make actions from a list of values, run them sequentially, and collect the results. `mapM`, a Prelude function, captures this pattern:

```
mapM :: (Monad m) => (a -> m b) -> [a] -> m [b]
```

We also have variants of the above functions with a trailing underscore in the name, such as `sequence_`, `mapM_` and `replicateM_`. These discard any final values and so are appropriate when you are only interested in performing actions. Compared with their underscore-less counterparts, these functions are like the distinction between `(>>)` and `(>>=)`. `mapM_` for instance has the following type:

```
mapM_ :: (Monad m) => (a -> m b) -> [a] -> m ()
```

Finally, it is worth mentioning that `Control.Monad` also provides `forM` and `forM_`, which are flipped versions of `mapM` and `mapM_`. `forM_` happens to be the idiomatic Haskell counterpart to the imperative for-each loop; and the type signature suggests that neatly:

```
forM_ :: (Monad m) => [a] -> (a -> m b) -> m ()
```

[10] `http://hackage.haskell.org/packages/archive/base/4.1.0.0/doc/html/Control-Monad.html`

Exercises:

1. Using the monadic functions we have just introduced, write a function which prints an arbitrary list of values.
2. Generalize the bunny invasion example[a] in the list monad chapter for an arbitrary number of generations.
3. What is the expected behavior of `sequence` for the `Maybe` monad?

[a] Chapter 32.2 on page 194

35 The State monad

If you have programmed in any other language before, you likely wrote some functions that "kept state". For those new to the concept, a *state* is one or more variables that are required to perform some computation but are not among the arguments of the relevant function. Object-oriented languages, like C++, suggest extensive use of state variables within objects in the form of member variables. Programs written in procedural languages, like C, typically use variables declared outside the current scope to keep track of state.

In Haskell, however, such techniques are not as straightforward to apply. They require mutable variables and imply functions will have hidden dependencies, which is at odds with Haskell's functional purity. Fortunately, in most cases it is possible to avoid such extra complications and keep track of state in a functionally pure way. We do so by passing the state information from one function to the next, thus making the hidden dependencies explicit. The `State` type is a tool crafted to make this process of threading state through functions more convenient. In this chapter, we will see how it can assist us in a typical problem involving state: generating pseudo-random numbers.

35.1 Pseudo-Random Numbers

Generating actual random numbers[1] is far from easy. Computer programs almost always use *pseudo*-random numbers[2] instead. They are called "pseudo" because they are not truly random. Rather, they are genererated by algorithms (the pseudo-random number generators) which take an initial state (commonly called the *seed*) and produce from it a sequence of numbers that have the appearance of being random.[3] Every time a pseudo-random number is requested, state somewhere must be updated, so that the generator can be ready for producing a fresh, different random number. Sequences of pseudo-random numbers can be replicated exactly if the initial seed and the generating algorithm are known.

35.1.1 Implementation in Haskell

Producing a pseudo-random number in most programming languages is very simple: there is a function somewhere in the libraries that provides a pseudo-random value (perhaps even a truly random one, depending on how it is implemented). Haskell has a similar one in the `System.Random` module from the `random` package:

[1] `https://en.wikibooks.org/wiki/%3Awikipedia%3ARandom%20number%20generation`

[2] `https://en.wikibooks.org/wiki/%3Awikipedia%3APseudorandom%20number%20generator`

[3] A common source of seeds is the current date and time as given by the internal clock of the computer. Assuming the clock is functioning correctly, it can provide unique seeds suitable for most day-to-day needs (as opposed to applications which demand high-quality randomness, as in cryptography or statistics).

```
GHCi> :m System.Random
GHCi> :t randomIO
randomIO :: Random a => IO a
GHCi> randomIO
-1557093684
GHCi> randomIO
1342278538
```

`randomIO` is an `IO` action. It couldn't be otherwise, as it makes use of mutable state, which is kept out of reach from our Haskell programs. Thanks to this hidden dependency, the pseudo-random values it gives back can be different every time.

35.1.2 Example: Rolling Dice

Figure 2 `randomRIO (1,6)`

Suppose we are coding a game in which at some point we need an element of chance. In real-life games that is often obtained by means of dice. So, let's create a dice-throwing function. We'll use the `IO` function `randomRIO`, which allows us to specify a range from which the pseudo-random values will be taken. For a 6 die, the call will be `randomRIO (1,6)`.

```
import Control.Applicative
import System.Random
```

```
rollDiceIO :: IO (Int, Int)
rollDiceIO = liftA2 (,) (randomRIO (1,6)) (randomRIO (1,6))
```

That function rolls two dice. Here, `liftA2` is used to make the two-argument function `(,)` work within a monad or applicative functor, in this case `IO`.[4] It can be easily defined in terms of `(<*>)`:

```
liftA2 f u v = f <$> u <*> v
```

As for `(,)`, it is the non-infix version of the tuple constructor. That being so, the two die rolls will be returned as a tuple in `IO`.

Exercises:

1. Implement a function `rollNDiceIO :: Int -> IO [Int]` that, given an integer (a number of die rolls), returns a list of that number of pseudo-random integers between 1 and 6.

35.1.3 Getting Rid of IO

A disadvantage of `randomIO` is that it requires us to use `IO` and store our state outside the program, where we can't control what happens to it. We would rather only use I/O when there is an unavoidable reason to interact with the outside world.

To avoid bringing `IO` into play, we can build a *local* generator. The `random` and `mkStdGen` functions in `System.Random` allow us to generate tuples containing a pseudo-random number together with an updated generator to use the next time the function is called.

```
GHCi> :m System.Random
GHCi> let generator = mkStdGen 0           -- "0" is our seed
GHCi> :t generator
generator :: StdGen
GHCi> generator
1 1
GHCi> :t random
random :: (RandomGen g, Random a) => g -> (a, g)
GHCi> random generator :: (Int, StdGen)
(2092838931,1601120196 1655838864)
```

Note:
In `random generator :: (Int, StdGen)`, we use the `::` to introduce a *type annotation*, which is essentially a type signature that we can put in the middle of an expression. Here, we are saying that the expression to the right, `random generator` has type `(Int, StdGen)`. It makes sense to use a type annotation here because, as we will discuss later, `random` can produce values of different types, so if we want it to give us an `Int` we'd better specify it in some way.

[4] If you need a refresher on applicative functors, have a look at the first section of the Prologue ^{https://en.wikibooks.org/wiki/Haskell%2FApplicative%20prologue%23Scene%201%3A%20Applicative} .

While we managed to avoid IO, there are new problems. First and foremost, if we want to use generator to get random numbers, the obvious definition...

```
GHCi> let randInt = fst . random $ generator :: Int
GHCi> randInt
2092838931
```

... is useless. It will always give back the same value, 2092838931, as the same generator in the same state will be used every time. To solve that, we can take the second member of the tuple (that is, the new generator) and feed it to a *new* call to random:

```
GHCi> let (randInt, generator') = random generator :: (Int, StdGen)
GHCi> randInt                             -- Same value
2092838931
GHCi> random generator' :: (Int, StdGen) -- Using new generator' returned from
 “random generator”
(-2143208520,439883729 1872071452)
```

That, of course, is clumsy and rather tedious, as we now need to deal with the fuss of carefully passing the generator around.

35.1.4 Dice without IO

We can re-do our dice throw with our new approach using the randomR function:

```
GHCi> randomR (1,6) (mkStdGen 0)
(6, 40014 40692)
```

The resulting tuple combines the result of throwing a single die with a new generator. A simple implementation for throwing two dice is then:

```
clumsyRollDice :: (Int, Int)
clumsyRollDice = (n, m)
        where
        (n, g) = randomR (1,6) (mkStdGen 0)
        (m, _) = randomR (1,6) g
```

Figure 3 Boxcars![a]

[a] http://en.wikipedia.org/wiki/Boxcars_%28slang%29

Exercises:

1. Implement a function `rollDice :: StdGen -> ((Int, Int), StdGen)` that, given a generator, return a tuple with our random numbers as first element and the last generator as the second.

The implementation of `clumsyRollDice` works as an one-off, but we have to manually pass the generator `g` from one `where` clause to the other. This approach becomes increasingly cumbersome as our programs get more complex, which means we have more values to shift around. It is also error-prone: what if we pass one of the middle generators to the wrong line in the `where` clause?

What we really need is a way to automate the extraction of the second member of the tuple (i.e. the new generator) and feed it to a new call to `random`. This is where the `State` comes into the picture.

35.2 Introducing State

Note:
In this chapter we will use the state monad provided by the module Control.Monad.Trans.State of the transformers package. By reading Haskell code in the wild, you will soon meet Control.Monad.State, a module of the closely related mtl package. The differences between these two modules need not concern us at the moment; everything we discuss here also applies to the mtl variant.

The Haskell type State describes *functions* that consume a state and produce both a result and an updated state, which are given back in a tuple.

The state function is wrapped by a data type definition which comes along with a runState accessor so that pattern matching becomes unnecessary. For our current purposes, the State type might be defined as:

```
newtype State s a = State { runState :: s -> (a, s) }
```

Here, s is the type of the state, and a the type of the produced result. Calling the type State is arguably a bit of a misnomer because the wrapped value is not the state itself but a *state processor*.

35.2.1 newtype

Note that we defined the data type with the newtype keyword, rather than the usual data. newtype can be used only for types with just one constructor and just one field. It ensures that the trivial wrapping and unwrapping of the single field is eliminated by the compiler. For that reason, simple wrapper types such as State are usually defined with newtype. Would defining a synonym with type be enough in such cases? Not really, because type does not allow us to define instances for the new data type, which is what we are about to do...

35.2.2 Where did the State constructor go?

When you start using Control.Monad.Trans.State, you will quickly notice there is no State constructor available. That was the reason for the ”for our current purposes” caveat a few paragraphs ago, when introducing the type. The transformers package implements the State type in a somewhat different way. The differences do not affect how we use or understand State; except that, instead of a State constructor, Control.Monad.Trans.State exports a state function,

```
state :: (s -> (a, s)) -> State s a
```

which does the same job. As for *why* the implementation is not the obvious one we presented above, we will get back to that a few chapters down the road.

35.2.3 Instantiating the Monad

So far, all we have done was to wrap a function type and give it a name. There is another ingredient, however: `State` is a monad, and that gives us very handy ways of using it. Unlike the instances of `Functor` or `Monad` we have seen so far, `State` has *two* type parameters. Since the type class only allows one parametrised parameter, the last one, we have to indicate the other one, `s`, will be fixed.

```
instance Monad (State s) where
```

That means there are actually *many* different `State` monads, one for each possible type of state - `State String`, `State Int`, `State SomeLargeDataStructure`, and so forth. Naturally, we only need to write one implementation of `return` and `(>>=)`; the methods will be able to deal with all choices of `s`.

The `return` function is implemented as:

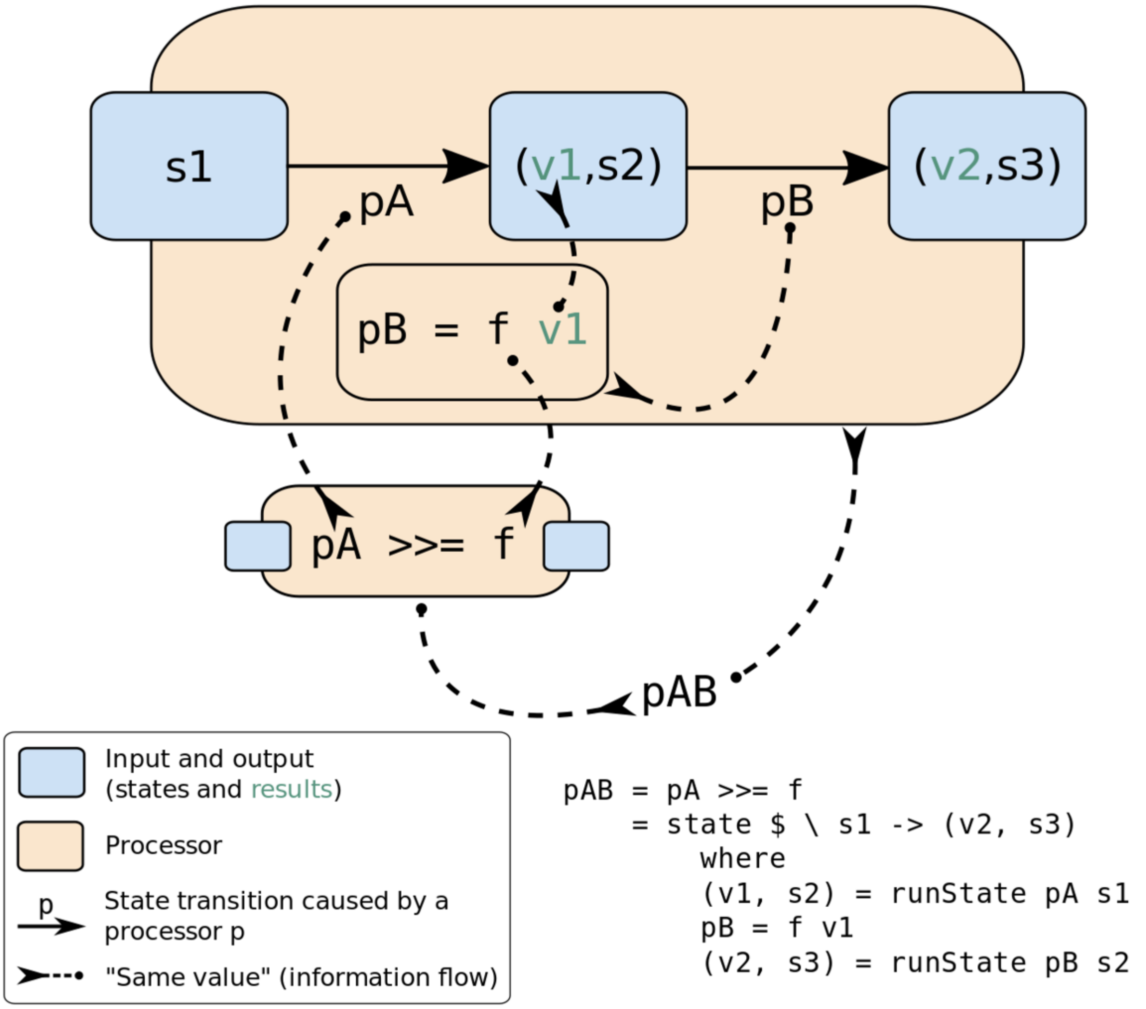
```
return :: a -> State s a
return x = state ( \ st -> (x, st) )
```

Giving a value (`x`) to `return` produces a function which takes a state (`st`) and returns it unchanged, together with value we want to be returned. As a finishing step, the function is wrapped up with the `state` function.

Binding is a bit intricate:

```
(>>=) :: State s a -> (a -> State s b) -> State s b
pr >>= k = state $ \ st ->
   let (x, st') = runState pr st -- Running the first processor on st.
   in runState (k x) st'       -- Running the second processor on st'.
```

`(>>=)` is given a state processor (`pr`) and a function (`k`) that is used to create another processor from the result of the first one. The two processors are combined into a function that takes the *initial* state (`st`) and returns the *second* result and the *third* state (i.e. the output of the second processor). Overall, `(>>=)` here allows us to run two state processors in sequence, while allowing the result of the first stage to influence what happens in the second one.



```
pAB = pA >>= f
    = state $ \ s1 -> (v2, s3)
        where
        (v1, s2) = runState pA s1
        pB = f v1
        (v2, s3) = runState pB s2
```

Figure 4 Schematic representation of how bind creates a new state processor (`pAB`) from a state processor (`pA`) and a processor-making function (`f`). `s1`, `s2` and `s3` are states. `v1` and `v2` are values. `pA`, `pB` and `pAB` are state processors. The wrapping and unwrapping by `state`/`runState` is implicit.

One detail in the implementation is how `runState` is used to undo the `State` wrapping, so that we can reach the function that will be applied to the states. The type of `runState pr`, for instance, is `s -> (a, s)`.

Another way to understand this derivation of the bind operator `>>=` is to consider once more the explicit but cumbersome way to simulate a stateful function of type `a -> b` by using functions of type `(a, s) -> (b, s)`, or, said another way: `a -> s -> (b,s) = a -> (s -> (b,s))`. These classes of functions pass the state on from function to function. Note that this last signature already suggests the right-hand side type in a bind operation where the abstract type `S b = (s -> (b, s))`.

Now that we have seen how the types seem to suggest the monadic signatures, lets consider a much more concrete question: Given two functions `f :: s -> (a, s)` and `g:: a -> s ->`

(b, s), how do we chain them to produce a new function that passes on the intermediate state?

This question does not require thinking about monads: one option is to simply use function composition. It helps our exposition if we just write it down explicitly as a lambda expression:

```
compose :: (s -> (a,s)) ->  {- first function -}
           a -> (s -> (b,s)) ->  {- second function,  note type is similar to
 (a,s) -> (b,s) -}
           s -> (b,s)  {- composed function -}
compose f g = \s0 -> let (a1, s1) = f s0 in (g a1) s1
{-This lambda expression threads both intermediate results produced by f into
 those required by g -}
```

Now, if in addition to chaining the input functions, we find that the functions of signature s -> (a,s) were all wrapped in an abstract datatype Wrapped a, and that therefore we need to call some other provided functionswrap :: (s -> (a,s)) -> Wrapped a, and unwrap :: Wrapped a -> (s -> (a,s)) in order to get to the inner function, then the code changes slightly:

```
{- what happens if the type  s -> (a,s) is wrapped and this new type is  called
 Wrapped a -}
composeWrapped :: Wrapped a -> (a -> Wrapped b) -> Wrapped b
composeWrapped wrappedf g =  wrap (\s0 -> let (a1,s1) = (unwrap wf) s0 in
 (unwrap (g a1)) s1)
```

This code is the implementation of (>>=) shown above, with wrap = state and unwrap = runState, so we can now see how the definition of bind given earlier is the standard function composition for this special kind of stateful function.

This explanation does not address yet where the original functions Wrapped a and a -> Wrapped b come from in the first place, but they do explain what you can do with them once you have them.

35.2.4 Setting and Accessing the State

The monad instance allows us to manipulate various state processors, but you may at this point wonder where exactly the *original* state comes from in the first place. That issue is handily dealt with by the function put:

```
put newState = state $ \_ -> ((), newState)
```

Given a state (the one we want to introduce), put generates a state processor which ignores whatever state it receives, and gives back the state we originally provided to put. Since we don't care about the result of this processor (all we want to do is to replace the state), the first element of the tuple will be (), the universal placeholder value.[5]

As a counterpart to put, there is get:

```
get = state $ \st -> (st, st)
```

[5] The technical term for both () and its type is *unit*.

The resulting state processor gives back the state st it is given in both as a result and as a state. That means the state will remain unchanged, and that a copy of it will be made available for us to manipulate.

35.2.5 Getting Values and State

As we have seen in the implementation of (>>=), runState is used to unwrap the State a b value to get the actual state processing function, which is then applied to some initial state. Other functions which are used in similar ways are evalState and execState. Given a State a b and an initial state, the function evalState will give back only the result value of the state processing, whereas execState will give back just the new state.

```
evalState :: State s a -> s -> a
evalState pr st = fst (runState pr st)

execState :: State s a -> s -> s
execState pr st = snd (runState pr st)
```

35.2.6 Dice and state

Time to use the State monad for our dice throw examples.

```
import Control.Monad.Trans.State
import System.Random
```

We want to generate Int dice throw results from a pseudo-random generator of type StdGen. Therefore, the type of our state processors will be State StdGen Int, which is equivalent to StdGen -> (Int, StdGen) bar the wrapping.

We can now implement a processor that, given a StdGen generator, produces a number between 1 and 6. Now, the type of randomR is:

```
-- The StdGen type we are using is an instance of RandomGen.
randomR :: (Random a, RandomGen g) => (a, a) -> g -> (a, g)
```

Doesn't it look familiar? If we assume a is Int and g is StdGen it becomes:

```
randomR (1, 6) :: StdGen -> (Int, StdGen)
```

We already have a state processing function! All that is missing is to wrap it with state:

```
rollDie :: State StdGen Int
rollDie = state $ randomR (1, 6)
```

For illustrative purposes, we can use get, put and do-notation to write rollDie in a very verbose way which displays explicitly each step of the state processing:

```
rollDie :: State StdGen Int
rollDie = do generator <- get
             let (value, newGenerator) = randomR (1,6) generator
             put newGenerator
             return value
```

Let's go through each of the steps:

1. First, we take out the pseudo-random generator from the monadic context with `<-`, so that we can manipulate it.
2. Then, we use the `randomR` function to produce an integer between 1 and 6 using the generator we took. We also store the new generator graciously returned by `randomR`.
3. We then set the state to be the `newGenerator` using `put`, so that any further `randomR` in the do-block, or further on in a `(>>=)` chain, will use a different pseudo-random generator.
4. Finally, we inject the result back into the `State StdGen` monad using `return`.

We can finally use our monadic die. As before, the initial generator state itself is produced by the `mkStdGen` function.

```
GHCi> evalState rollDie (mkStdGen 0)
6
```

Why have we involved monads and built such an intricate framework only to do exactly what `fst $ randomR (1,6)` already does? Well, consider the following function:

```
rollDice :: State StdGen (Int, Int)
rollDice = liftA2 (,) rollDie rollDie
```

We obtain a function producing *two* pseudo-random numbers in a tuple. Note that these are in general different:

```
GHCi> evalState rollDice (mkStdGen 666)
 (6,1)
```

Under the hood, state is being passed through `(>>=)` from one `rollDie` computation to the other. Doing that was previously very clunky using `randomR (1,6)` alone because we had to pass state manually. Now, the monad instance is taking care of that for us. Assuming we know how to use the lifting functions, constructing intricate combinations of pseudo-random numbers (tuples, lists, whatever) has suddenly become much easier.

Exercises:

1. Similarly to what was done for `rollNDiceIO`, implement a function `rollNDice :: Int -> State StdGen [Int]` that, given an integer, returns a list with that number of pseudo-random integers between 1 and 6.
2. Write an instance of `Functor` for `State s`. Your final answer should not use anything that mentions `Monad` in its type (that is, `return`, `(>>=)`, etc.). Then, explain in a few words what the `fmap` you wrote does.(Hint: If you get stuck, have another look at the comments about `liftM` at the very end of Understanding monads[a].)
3. Besides `put` and `get`, there are also
   `modify :: (s -> s) -> State s ()`
   which modifies the current state using a function, and
   `gets :: (s -> a) -> State s a`
   which produces a modified copy of the state while leaving the state itself unchanged. Write implementations for them.

[a] Chapter 30.5 on page 186

35.3 Pseudo-random values of different types

Until now, we have used only `Int` as type of the value produced by the pseudo-random generator. However, looking at the type of `randomR` shows we are not restricted to `Int`. It can generate values of any type in the `Random` class from System.Random[6]. There already are instances for `Int`, `Char`, `Integer`, `Bool`, `Double` and `Float`, so you can immediately generate any of those.

Because `State StdGen` is ”agnostic” in regard to the type of the pseudo-random value it produces, we can write a similarly ”agnostic” function that provides a pseudo-random value of unspecified type (as long as it is an instance of `Random`):

```
getRandom :: Random a => State StdGen a
getRandom = state random
```

Compared to `rollDie`, this function does not specify the `Int` type in its signature and uses `random` instead of `randomR`; otherwise, it is just the same. `getRandom` can be used for any instance of `Random`:

```
GHCi> evalState getRandom (mkStdGen 0) :: Bool
True
GHCi> evalState getRandom (mkStdGen 0) :: Char
'\64685'
GHCi> evalState getRandom (mkStdGen 0) :: Double
0.9872770354820595
GHCi> evalState getRandom (mkStdGen 0) :: Integer
2092838931
```

Indeed, it becomes quite easy to conjure all these at once:

[6] http://hackage.haskell.org/packages/archive/random/latest/doc/html/System-Random.html

```
someTypes :: State StdGen (Int, Float, Char)
someTypes = liftA3 (,,) getRandom getRandom getRandom

allTypes :: State StdGen (Int, Float, Char, Integer, Double, Bool, Int)
allTypes = (,,,,,,) <$> getRandom
                    <*> getRandom
                    <*> getRandom
                    <*> getRandom
                    <*> getRandom
                    <*> getRandom
                    <*> getRandom
```

For writing `allTypes`, there is no `liftA7`,[7] and so we resort to plain old `(<*>)` instead. Using it, we can apply the tuple constructor to each of the seven random values in the `State StdGen` monadic context.

`allTypes` provides pseudo-random values for all default instances of `Random`; an additional `Int` is inserted at the end to prove that the generator is not the same, as the two `Int`s will be different.

```
GHCi> evalState allTypes (mkStdGen 0)
GHCi>
(2092838931,9.953678e-4,'\825586',-868192881,0.4188001483955421,False,316817438)
```

Exercises:

1. If you are not convinced that `State` is worth using, try to implement a function equivalent to `evalState allTypes` without making use of monads, i.e. with an approach similar to `clumsyRollDice` above.

[7] Beyond `liftA3`, the standard libraries only provide the monad-only `liftM4` and `liftM5` in `Control.Monad`.

36 Alternative and MonadPlus

In our studies so far, we saw that both Maybe and lists can represent computations with a varying number of results. We use Maybe to indicate a computation can fail somehow (that is, it can have either zero results or one result), and we use lists for computations that can have many possible results (ranging from zero to arbitrarily many results). In both of these cases, one useful operation is amalgamating *all* possible results from multiple computations into a single computation. With lists, for instance, that would amount to concatenating lists of possible results. The Alternative class captures this amalgamation in a general way.

36.1 Definition

Note:
The Alternative class and its methods can be found in the Control.Applicative[a] module.

[a] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Applicative.html

Alternative is a subclass of Applicative whose instances must define, at a minimum, the following two methods:

```
class Applicative f => Alternative f where
  empty :: f a
  (<|>) :: f a -> f a -> f a
```

empty is an applicative computation with zero results, while (<|>) is a binary function which combines two computations.

Here are the two instance definitions for Maybe and lists:

```
instance Alternative Maybe where
  empty               = Nothing
  -- Note that this could have been written more compactly.
  Nothing <|> Nothing = Nothing -- 0 results + 0 results = 0 results
  Just x  <|> Nothing = Just x  -- 1 result  + 0 results = 1 result
  Nothing <|> Just x  = Just x  -- 0 results + 1 result  = 1 result
  Just x  <|> Just y  = Just x  -- 1 result  + 1 result  = 1 result:
                                -- Maybe can only hold up to one result,
                                -- so we discard the second one.

instance Alternative [] where
  empty = []
  (<|>) = (++) -- length xs + length ys = length (xs ++ ys)
```

36.2 Example: parallel parsing

Traditional input parsing involves functions which consume an input one character at a time. That is, a parsing function takes an input string and chops off (i.e. "consumes") characters from the front if they satisfy certain criteria. For example, you could write a function which consumes one uppercase character. If the characters on the front of the string don't satisfy the given criteria, the parser has *failed*. In the example below, for instance, we consume a digit in the input and return the digit that was parsed. The possibility of failure is expressed by using `Maybe`.

```
digit :: Int -> String -> Maybe Int
digit i s | i > 9 || i < 0 = Nothing
          | otherwise      = do
  let (c:_) = s
  if [c] == show i then Just i else Nothing
```

The guards assure that the `Int` we are checking for is a single digit. Otherwise, we are just checking that the first character of our String matches the digit we are checking for. If it passes, we return the digit wrapped in a `Just`. The do-block assures that any failed pattern match will result in returning `Nothing`.

Now, `(<|>)` can be used to run two parsers *in parallel*. That is, we use the result of the first one if it succeeds, and otherwise, we use the result of the second. If both fail, then the combined parser returns `Nothing`. We can use `digit` with `(<|>)` to, for instance, parse strings of binary digits:

```
binChar :: String -> Maybe Int
binChar s = digit 0 s <|> digit 1 s
```

Parser libraries often make use of `Alternative` in this way. Two examples are `(+++)` in Text.ParserCombinators.ReadP[1] and `(<|>)` in Text.ParserCombinators.Parsec.Prim[2]. This usage pattern can be described in terms of *choice*. For instance, if we want to give `binChar` a string that will be successfully parsed, we have two choices: either to begin the string with `'0'` or with `'1'`.

36.3 MonadPlus

`MonadPlus` is a class which is closely related to `Alternative`:

```
class Monad m => MonadPlus m where
  mzero :: m a
  mplus :: m a -> m a -> m a
```

This definition is exactly like that of `Alternative`, only with different method names and the `Applicative` constraint being changed into `Monad`. Unsurprisingly, for types that have instances of both `Alternative` and `MonadPlus`, `mzero` and `mplus` should be equivalent to `empty` and `(<|>)` respectively.

[1] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Text-ParserCombinators-ReadP.html

[2] http://hackage.haskell.org/packages/archive/parsec/latest/doc/html/Text-ParserCombinators-Parsec-Prim.h

One might legitimately wonder why the seemingly redundant MonadPlus class exists. Part of the reason is historical: just like Monad existed in Haskell long before Applicative was introduced, MonadPlus is much older than Alternative. Beyond such accidents, there are also additional expectations about how the MonadPlus methods should interact with the Monad ones that do not apply to Alternative, and so saying something is a MonadPlus is a stronger claim than saying it is both an Alternative and a Monad. We will make some additional considerations about this issue in the following section.

36.4 Alternative and MonadPlus laws

Like most general-purpose classes Alternative and MonadPlus are expected to follow a handful of laws. However, there isn't universal agreement on what the full set of laws should look like. The most commonly adopted laws, and the most crucial for providing intuition about Alternative say that empty and (<|>) form a *monoid*. By that, we mean:

```
-- empty is a neutral element
empty <|> u  =  u
u <|> empty  =  u
-- (<|>) is associative
u <|> (v <|> w)  =  (u <|> v) <|> w
```

There is nothing fancy about ”forming a monoid”: in the above, ”neutral element” and ”associative” here is just like how addition of integer numbers is said to be associative and to have zero as neutral element. In fact, this analogy is the source of the MonadPlus methods, mzero and mplus.

As for MonadPlus, at a minimum there usually are the monoid laws, which correspond exactly to the ones just above...

```
mzero `mplus` m  =  m
m `mplus` mzero  =  m
m `mplus` (n `mplus` o)  =  (m `mplus` n) `mplus` o
```

... plus the additional two laws, quoted by the Control.Monad[3] documentation:

```
mzero >>= f  =  mzero -- left zero
m >> mzero   =  mzero -- right zero
```

If mzero is interpreted as a failed computation, these laws state that a failure within a chain of monadic computations leads to the failure of the whole chain.

We will touch upon some additional suggestions of laws for Alternative and MonadPlus at the end of the chapter.

36.5 Useful functions

In addition to (<|>) and empty, there are two other general-purpose functions in the base libraries involving Alternative.

[3] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Monad.html

36.5.1 asum

A common task when working with `Alternative` is taking a list of alternative values, e.g. `[Maybe a]` or `[[a]]`, and folding it down with `(<|>)`. The function `asum`, from `Data.Foldable` fulfills this role:

```
asum :: (Alternative f, Foldable t) => t (f a) -> f a
asum = foldr (<|>) empty
```

In a sense, `asum` generalizes the list-specific `concat` operation. Indeed, the two are equivalent when the lists are the `Alternative` being used. For Maybe, `asum` finds the first `Just x` in the list and returns `Nothing` if there aren't any.

It should also be mentioned that `msum`, available from both 'Data.Foldable' and 'Control.Monad', is just `asum` specialised to `MonadPlus`.

```
msum :: (MonadPlus m, Foldable t) => t (m a) -> m a
```

36.5.2 guard

When discussing the list monad[4] we noted how similar it was to list comprehensions, but we didn't discuss how to mirror list comprehension filtering. The `guard` function from `Control.Monad` allows us to do exactly that.

Consider the following comprehension which retrieves all pythagorean triples[5] (i.e. trios of integer numbers which work as the lengths of the sides for a right triangle). First we'll examine the brute-force approach. We'll use a boolean condition for filtering; namely, Pythagoras' theorem:

```
pythags = [ (x, y, z) | z <- [1..], x <- [1..z], y <- [x..z], x^2 + y^2 == z^2 ]
```

The translation of the comprehension above to a list monad do-block is:

```
pythags = do
  z <- [1..]
  x <- [1..z]
  y <- [x..z]
  guard (x^2 + y^2 == z^2)
  return (x, y, z)
```

The `guard` function can be defined for all `Alternative`s like this:

```
guard :: Alternative m => Bool -> m ()
guard True  = return ()
guard _ = empty
```

`guard` will reduce a do-block to `empty` if its predicate is `False`. Given the left zero law...

```
mzero >>= f = mzero
-- Or, equivalently:
empty >>= f = empty
```

[4] Chapter 32 on page 193

[5] https://en.wikipedia.org/wiki/Pythagorean_triple

... an `empty` on the left-hand side of an `>>=` operation will produce `empty` again. As do-blocks are decomposed to lots of expressions joined up by `(>>=)`, an `empty` at any point will cause the entire do-block to become `empty`.

Let's examine in detail what `guard` does in the `pythags`. First, here is `guard` defined for the list monad:

```
-- guard :: Bool -> [()]
guard True  = [()]
guard _ = []
```

Basically, `guard` *blocks off* a route. In `pythags`, we want to block off all the routes (or combinations of `x`, `y` and `z`) where `x^2 + y^2 == z^2` is `False`. Let's look at the expansion of the above `do`-block to see how it works:

```
pythags =
  [1..] >>= \z ->
  [1..z] >>= \x ->
  [x..z] >>= \y ->
  guard (x^2 + y^2 == z^2) >>= \_ ->
  return (x, y, z)
```

Replacing `>>=` and `return` with their definitions for the list monad (and using some let-bindings to keep it readable), we obtain:

```
pythags =
 let ret x y z = [(x, y, z)]
     gd  z x y = concatMap (\_ -> ret x y z) (guard $ x^2 + y^2 == z^2)
     doY z x   = concatMap (gd  z x) [x..z]
     doX z     = concatMap (doY z  ) [1..z]
     doZ       = concatMap (doX    ) [1..]
 in doZ
```

Remember that `guard` returns the empty list in the case of its argument being `False`. Mapping across the empty list produces the empty list, no matter what function you pass in. So an empty list produced by the call to `guard` in `gd` will cause `gd` to produce an empty list, with `\_ -> ret x y z`, which would otherwise add a result, not being actually called.

To understand why this matters, think about list-computations as a tree. With our Pythagorean triple algorithm, we need a branch starting from the top for every choice of `z`, then a branch from each of these branches for every value of `x`, then from each of these, a branch for every value of `y`. So the tree looks like this:

```
     start
     |_________________________...
     |    |         |
  z  1    2         3
     |    |____     |____________
     |    |    |    |       |    |
  x  1    1    2    1       2    3
     |    |_   |    |___    |_   |
     |    | |  |    | | |   | |  |
  y  1    1 2  2    1 2 3   2 3  3
```

Each combination of z, x and y represents a route through the tree. Once all the functions have been applied, the results of each branch are concatenated together, starting from the bottom. Any route where our predicate doesn't hold evaluates to an empty list, and so has no impact on this concatenation.

36.6 Exercises

Exercises:

1. Prove the `Alternative` monoid laws for `Maybe` and lists.
2. We could augment the parser from the parallel parsing example so that it would handle any character, in the following manner:

```
-- | Consume a given character in the input, and return
--   the character we just consumed, paired with rest of
--   the string. We use a do-block  so that if the
--   pattern match fails at any point, 'fail' of the
--   Maybe monad (i.e. Nothing) is returned.
char :: Char -> String -> Maybe (Char, String)
char c s = do
  let (c':s') = s
  if c == c' then Just (c, s') else Nothing
```

   It would then be possible to write a `hexChar` function which parses any valid hexadecimal character (0-9 or a-f). Try writing this function (hint: `map digit [0..9] :: [String -> Maybe Int]`).
3. Use `guard` and the `Applicative` combinators (`pure`, `(<*>)`, `(*>)`, etc.) to implement `safeLog` from the Maybe monad chapter[a]. Do not use the `Monad` combinators (`return`, `(>>=)`, `(>>)`, etc.).

[a] Chapter 31 on page 189

36.7 Relationship with monoids

When discussing the `Alternative` laws, we alluded to the mathematical concept of monoids. It turns out that there is a `Monoid` class in Haskell, defined in Data.Monoid[6]. A fuller presentation of will be given in a later chapter[7]. For now, it suffices to say that a minimal definition of `Monoid` implements two methods; namely, a neutral element (or 'zero') and an associative binary operation (or 'plus').

```
class Monoid m where
  mempty  :: m
  mappend :: m -> m -> m
```

For example, lists form a simple monoid:

[6] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data.Monoid.html

[7] Chapter 39 on page 241

```
instance Monoid [a] where
  mempty  = []
  mappend = (++)
```

Looks familiar, doesn't it? In spite of the uncanny resemblance to `Alternative` and `MonadPlus`, there is a key difference. Note the use of `[a]` instead of `[]` in the instance declaration. Monoids are not necessarily ”wrappers” of anything, or parametrically polymorphic. For instance, the integer numbers on form a monoid under addition with `0` as neutral element. `Alternative` is a separate type class because it captures a specific sort of monoid with distinctive properties – for instance, a binary operation (< that is intrinsically linked to an `Applicative` context.

36.8 Other suggested laws

Note:
Consider this as a bonus section. While it is good to be aware of there being various takes on these laws, the whole issue is, generally speaking, not one worth losing sleep over.

Beyond the commonly assumed laws mentioned a few sections above, there are a handful of others which make sense from certain perspectives, but do not hold for all existing instances of `Alternative` and `MonadPlus`. The current `MonadPlus`, in particular, might be seen as an intersection between a handful of hypothetical classes that would have additional laws.

The following two additional laws are commonly suggested for `Alternative`. While they do hold for both `Maybe` and lists, there are counterexamples in the core libraries. Also note that, for `Alternative`s that are also `MonadPlus`, the `mzero` laws mentioned earlier are not a consequence of these laws.

```
(f <|> g) <*> a = (f <*> a) <|> (g <*> a) -- right distributivity (of <*>)
empty <*> a = empty -- right absorption (for <*>)
```

As for `MonadPlus`, a common suggestion is the *left distribution* law, which holds for lists, but not for `Maybe`:

```
(m `mplus` n) >>= k  =  (m >>= k) `mplus` (n >>= k) -- left distribution
```

Conversely, the *left catch* law holds for `Maybe` but not for lists:

```
return x `mplus` m = return m -- left catch
```

It is generally assumed that at least one of left distribution and left catch will hold for any `MonadPlus` instance.

Finally, it is worth noting that there are divergences even about the monoid laws. One case sometimes raised against them is that for certain non-determinism monads typically expressed in terms of `MonadPlus` the key laws are left zero and left distribution, while the monoid laws in such cases lead to difficulties and should be relaxed or dropped entirely.

Some entirely optional further reading, for the curious reader:

- The Haskell Wiki on MonadPlus[8] (note that this debate long predates the existence of `Alternative`).
- *Distinction between typeclasses MonadPlus, Alternative, and Monoid?*[9] and *Confused by the meaning of the 'Alternative' type class and its relationship to other type classes*[10] at Stack Overflow (detailed overviews of the *status quo* reflected by the documentation of the relevant libraries as of GHC 7.x/8.x – as opposed to the 2010 Haskell Report, which is less prescriptive on this matter.)
- *From monoids to near-semirings: the essence of MonadPlus and Alternative* by Rivas, Jaskelioff and Schrijvers[11] (a formulation that includes, beyond the monoid laws, right distribution and right absorption for `Alternative`, as well as left zero and left distribution for `MonadPlus`).
- Wren Romano on MonadPlus and seminearrings[12] (argues that the `MonadPlus` right zero law is too strong).
- Oleg Kiselyov on the MonadPlus laws[13] (argues against the monoid laws in the case of non-determinism monads).
- *Must mplus always be associative?* at Stack Overflow[14] (a discussion about the merits of the monoid laws of `MonadPlus`).

[8] http://www.haskell.org/haskellwiki/MonadPlus

[9] http://stackoverflow.com/q/10167879/2751851

[10] http://stackoverflow.com/q/13080606/2751851

[11] https://lirias.kuleuven.be/handle/123456789/499951

[12] http://winterkoninkje.dreamwidth.org/90905.html

[13] http://okmij.org/ftp/Computation/monads.html#monadplus

[14] http://stackoverflow.com/q/15722906/2751851

37 Monad transformers

We have seen how monads can help handling IO actions, Maybe, lists, and state. With monads providing a common way to use such useful general-purpose tools, a natural thing we might want to do is using the capabilities of *several* monads at once. For instance, a function could use both I/O and Maybe exception handling. While a type like IO (Maybe a) would work just fine, it would force us to do pattern matching within IO do-blocks to extract values, something that the Maybe monad was meant to spare us from.

Enter **monad transformers**: special types that allow us to roll two monads into a single one that shares the behavior of both.

37.1 Passphrase validation

Consider a real-life problem for IT staff worldwide: getting users to create strong passphrases. One approach: force the user to enter a minimum length with various irritating requirements (such as at least one capital letter, one number, one non-alphanumeric character, etc.)

Here's a Haskell function to acquire a passphrase from a user:

```
getPassphrase :: IO (Maybe String)
getPassphrase = do s <- getLine
                   if isValid s then return $ Just s
                                else return Nothing

-- The validation test could be anything we want it to be.
isValid :: String -> Bool
isValid s = length s >= 8
            && any isAlpha s
            && any isNumber s
            && any isPunctuation s
```

First and foremost, getPassphrase is an IO action, as it needs to get input from the user. We also use Maybe, as we intend to return Nothing in case the password does not pass the isValid. Note, however, that we aren't actually using Maybe as a monad here: the do block is in the IO monad, and we just happen to return a Maybe value into it.

Monad transformers not only make it easier to write getPassphrase but also simplify all the code instances. Our passphrase acquisition program could continue like this:

```
askPassphrase :: IO ()
askPassphrase = do putStrLn "Insert your new passphrase:"
                   maybe_value <- getPassphrase
                   if isJust maybe_value
                     then do putStrLn "Storing in database..." -- do stuff
                     else putStrLn "Passphrase invalid."
```

The code uses one line to generate the `maybe_value` variable followed by further validation of the passphrase.

With monad transformers, we will be able to extract the passphrase in one go — without any pattern matching or equivalent bureaucracy like `isJust`. The gains for our simple example might seem small but will scale up for more complex situations.

37.2 A simple monad transformer: `MaybeT`

To simplify `getPassphrase` and all the code that uses it, we will define a *monad transformer* that gives the `IO` monad some characteristics of the `Maybe` monad; we will call it `MaybeT`. That follows a convention where monad transformers have a ”T” appended to the name of the monad whose characteristics they provide.

`MaybeT` is a wrapper around `m (Maybe a)`, where `m` can be any monad (`IO` in our example):

```
newtype MaybeT m a = MaybeT { runMaybeT :: m (Maybe a) }
```

This data type definition specifies a `MaybeT` type constructor, parameterized over `m`, with a term constructor, also called `MaybeT`, and a convenient accessor function `runMaybeT`, with which we can access the underlying representation.

The whole point of monad transformers is that *they are monads themselves*; and so we need to make `MaybeT m` an instance of the `Monad` class:

```
instance Monad m => Monad (MaybeT m) where
    return  = MaybeT . return . Just
```

It would also have been possible (though arguably less readable) to write `return = MaybeT . return . return`.

As in all monads, the bind operator is the heart of the transformer.

```
-- The signature of (>>=), specialized to MaybeT m
(>>=) :: MaybeT m a -> (a -> MaybeT m b) -> MaybeT m b

x >>= f = MaybeT $ do maybe_value <- runMaybeT x
                      case maybe_value of
                          Nothing    -> return Nothing
                          Just value -> runMaybeT $ f value
```

Starting from the first line of the `do` block:

- First, the `runMaybeT` accessor unwraps `x` into an `m (Maybe a)` computation. That shows us that the whole `do` block is in `m`.
- Still in the first line, `<-` extracts a `Maybe a` value from the unwrapped computation.
- The `case` statement tests `maybe_value`:
  - With `Nothing`, we return `Nothing` into `m`;
  - With `Just`, we apply `f` to the `value` from the Just. Since `f` has `MaybeT m b` as result type, we need an extra `runMaybeT` to put the result back into the `m` monad.
- Finally, the `do` block as a whole has `m (Maybe b)` type; so it is wrapped with the `MaybeT` constructor.

It may look a bit complicated; but aside from the copious amounts of wrapping and unwrapping, the implementation does the same as the familiar bind operator of `Maybe`:

```
-- (>>=) for the Maybe monad
maybe_value >>= f = case maybe_value of
                        Nothing -> Nothing
                        Just value -> f value
```

Why use the `MaybeT` constructor before the `do` block while we have the accessor `runMaybeT` within `do`? Well, the `do` block must be in the `m` monad, not in `MaybeT m` (which lacks a defined bind operator at this point).

Note:
The chained functions in the definition of `return` suggest a metaphor, which you may find either useful or confusing. Consider the combined monad as a *sandwich*. This metaphor might suggest three layers of monads in action, but there are only two really: the inner monad and the combined monad (there are no binds or returns done in the base monad; it only appears as part of the implementation of the transformer). If you like this metaphor at all, think of the transformer and the base monad as two parts of the same thing - the *bread* - which wraps the inner monad.

Technically, this is all we need; however, it is convenient to make `MaybeT` an instance of a few other classes:

```
instance Monad m => Alternative (MaybeT m) where
    empty   = MaybeT $ return Nothing
    x <|> y = MaybeT $ do maybe_value <- runMaybeT x
                          case maybe_value of
                               Nothing    -> runMaybeT y
                               Just _     -> return maybe_value

instance Monad m => MonadPlus (MaybeT m) where
    mzero = empty
    mplus = (<|>)

instance MonadTrans MaybeT where
    lift = MaybeT . (liftM Just)
```

`MonadTrans` implements the `lift` function, so we can take functions from the `m` monad and bring them into the `MaybeT m` monad in order to use them in `do` blocks. As for `Alternative` and `MonadPlus`, since `Maybe` is an instance of those class it makes sense to make the `MaybeT` an instance too.

37.2.1 Application to the passphrase example

With all this done, here is what the previous example of passphrase management looks like:

```
getValidPassphrase :: MaybeT IO String
getValidPassphrase = do s <- lift getLine
                        guard (isValid s) -- Alternative provides guard.
                        return s

askPassphrase :: MaybeT IO ()
askPassphrase = do lift $ putStrLn "Insert your new passphrase:"
```

```
                  value <- getValidPassphrase
                  lift $ putStrLn "Storing in database..."
```

The code is now simpler, especially in the user function `askPassphrase`. Most importantly, we do not have to manually check whether the result is `Nothing` or `Just`: the bind operator takes care of that for us.

Note how we use `lift` to bring the functions `getLine` and `putStrLn` into the `MaybeT IO` monad. Also, since `MaybeT IO` is an instance of `Alternative`, checking for passphrase validity can be taken care of by a `guard` statement, which will return `empty` (i.e. `IO Nothing`) in case of a bad passphrase.

Incidentally, with the help of `MonadPlus` it also becomes very easy to ask the user *ad infinitum* for a valid passphrase:

```
askPassword :: MaybeT IO ()
askPassword = do lift $ putStrLn "Insert your new password:"
                 value <- msum $ repeat getValidPassphrase
                 lift $ putStrLn "Storing in database..."
```

37.3 A plethora of transformers

The `transformers` package provides modules with transformers for many common monads (`MaybeT`, for instance, can be found in Control.Monad.Trans.Maybe[1]). These are defined consistently with their non-transformer versions; that is, the implementation is basically the same except with the extra wrapping and unwrapping needed to thread the other monad. From this point on, we will use **base monad** to refer to the non-transformer monad (e.g. Maybe in MaybeT) on which a transformer is based and **inner monad** to refer to the other monad (e.g. IO in MaybeT IO) on which the transformer is applied.

To pick an arbitrary example, `ReaderT Env IO String` is a computation which involves reading values from some environment of type `Env` (the semantics of `Reader`, the base monad) and performing some `IO` in order to give a value of type `String`. Since the bind operator and `return` for the transformer mirror the semantics of the base monad, a `do` block of type `ReaderT Env IO String` will, from the outside, look a lot like a `do` block of the `Reader` monad, except that `IO` actions become trivial to embed by using `lift`.

37.3.1 Type juggling

We have seen that the type constructor for `MaybeT` is a wrapper for a `Maybe` value in the inner monad. So, the corresponding accessor `runMaybeT` gives us a value of type `m (Maybe a)` - i.e. a value of the base monad returned in the inner monad. Similarly, for the `ListT` and `ExceptT` transformers, which are built around lists and `Either` respectively:

```
runListT :: ListT m a -> m [a]
```

and

[1] http://hackage.haskell.org/packages/archive/transformers/latest/doc/html/Control-Monad-Trans-Maybe.html

```
runExceptT :: ExceptT e m a -> m (Either e a)
```

Not all transformers are related to their base monads in this way, however. Unlike the base monads in the two examples above, the `Writer`, `Reader`, `State`, and `Cont` monads have neither multiple constructors nor constructors with multiple arguments. For that reason, they have `run...` functions which act as simple unwrappers, analogous to the `run...T` of the transformer versions. The table below shows the result types of the `run...` and `run...T` functions in each case, which may be thought of as the types wrapped by the base and transformed monads respectively.[2]

Base Monad	**Transformer**	**Original Type ("wrapped" by base)**	**Combined Type ("wrapped" by transformer)**
Writer	WriterT	`(a, w)`	`m (a, w)`
Reader	ReaderT	`r -> a`	`r -> m a`
State	StateT	`s -> (a, s)`	`s -> m (a, s)`
Cont	ContT	`(a -> r) -> r`	`(a -> m r) -> m r`

Notice that the base monad is absent in the combined types. Without interesting constructors (of the sort for `Maybe` or lists), there is no reason to retain the base monad type after unwrapping the transformed monad. It is also worth noting that in the latter three cases we have function types being wrapped. `StateT`, for instance, turns state-transforming functions of the form `s -> (a, s)` into state-transforming functions of the form `s -> m (a, s)`; only the result type of the wrapped function goes into the inner monad. `ReaderT` is analogous.`ContT` is different because of the semantics of `Cont` (the *continuation* monad): the result types of both the wrapped function and its function argument must be the same, and so the transformer puts both into the inner monad. In general, there is no magic formula to create a transformer version of a monad; the form of each transformer depends on what makes sense in the context of its non-transformer type.

37.4 Lifting

We will now have a more detailed look at the `lift` function, which is critical in day-to-day use of monad transformers. The first thing to clarify is the name "lift". One function with a similar name that we already know is `liftM`. As we have seen in Understanding monads[3], it is a monad-specific version of `fmap`:

```
liftM :: Monad m => (a -> b) -> m a -> m b
```

`liftM` applies a function `(a -> b)` to a value within a monad `m`. We can also look at it as a function of just one argument:

```
liftM :: Monad m => (a -> b) -> (m a -> m b)
```

[2] The wrapping interpretation is only literally true for versions of the `mtl` package older than 2.0.0.0 .

[3] Chapter 30.5 on page 186

liftM converts a plain function into one that acts within m. By "lifting", we refer to bringing something into something else — in this case, a function into a monad.

liftM allows us to apply a plain function to a monadic value without needing do-blocks or other such tricks:

do notation

```
do x <- monadicValue
   return (f x)
```

liftM

```
liftM f monadicValue
```

The lift function plays an analogous role when working with monad transformers. It brings (or, to use another common word for that, *promotes*) inner monad computations to the combined monad. By doing so, it allows us to easily insert inner monad computations as part of a larger computation in the combined monad.

lift is the single method of the MonadTrans class, found in Control.Monad.Trans.Class[4]. All monad transformers are instances of MonadTrans, and so lift is available for them all.

```
class MonadTrans t where
    lift :: (Monad m) => m a -> t m a
```

There is a variant of lift specific to IO operations, called liftIO, which is the single method of the MonadIO class in Control.Monad.IO.Class[5].

```
class (Monad m) => MonadIO m where
   liftIO :: IO a -> m a
```

liftIO can be convenient when multiple transformers are stacked into a single combined monad. In such cases, IO is always the innermost monad, and so we typically need more than one lift to bring IO values to the top of the stack. liftIO is defined for the instances in a way that allows us to bring an IO value from any depth while writing the function a single time.

37.4.1 Implementing lift

Implementing lift is usually pretty straightforward. Consider the MaybeT transformer:

```
instance MonadTrans MaybeT where
    lift m = MaybeT (liftM Just m)
```

We begin with a monadic value of the inner monad. With liftM (fmap would have worked just as fine), we slip the base monad (through the Just constructor) underneath, so that we go from m a to m (Maybe a)). Finally, we use the MaybeT constructor to wrap up the monadic sandwich. Note that the liftM here works in the inner monad, just like the do-block wrapped by MaybeT in the implementation of (>>=) we saw early on was in the inner monad.

[4] http://hackage.haskell.org/packages/archive/transformers/latest/doc/html/Control-Monad-Trans-Class.html

[5] http://hackage.haskell.org/packages/archive/transformers/latest/doc/html/Control-Monad-IO-Class.html

Exercises:

1. Why is it that the `lift` function has to be defined separately for each monad, where as `liftM` can be defined in a universal way?
2. `Identity` is a trivial functor, defined in `Data.Functor.Identity` as:
   `newtype Identity a = Identity { runIdentity :: a }`
   It has the following `Monad` instance:

   ```
   instance Monad Identity where
       return a = Identity a
       m >>= k  = k (runIdentity m)
   ```

   Implement a monad transformer `IdentityT`, analogous to `Identity` but wrapping values of type `m a` rather than `a`. Write at least its `Monad` and `MonadTrans` instances.

37.5 Implementing transformers

37.5.1 The State transformer

As an additional example, we will now have a detailed look at the implementation of `StateT`. You might want to review the section on the State monad[6] before continuing.

Just as the State monad might have been built upon the definition `newtype State s a = State { runState :: (s -> (a,s)) }`, the StateT transformer is built upon the definition:

```
newtype StateT s m a = StateT { runStateT :: (s -> m (a,s)) }
```

`StateT s m` will have the following `Monad` instance, here shown alongside the one for the base state monad:

State

```
newtype State s a =
  State { runState :: (s -> (a,s)) }

instance Monad (State s) where
  return a        = State $ \s -> (a,s)
  (State x) >>= f = State $ \s ->
    let (v,s') = x s
    in runState (f v) s'
```

StateT

```
newtype StateT s m a =
  StateT { runStateT :: (s -> m (a,s)) }

instance (Monad m) => Monad (StateT s m) where
  return a         = StateT $ \s -
> return (a,s)
  (StateT x) >>= f = StateT $ \s -> do
    (v,s') <- x s          --
 get new value and state
    runStateT (f v) s'     -- pass them to f
```

Our definition of `return` makes use of the `return` function of the inner monad. `(>>=)` uses a do-block to perform a computation in the inner monad.

[6] Chapter 35 on page 207

Note:
Incidentally, we can now finally explain why, back in the chapter about `State`[a], there was a `state` function instead of a `State` constructor. In the `transformers` and `mtl` packages, `State s` is implemented as a type synonym for `StateT s Identity`, with `Identity` being the dummy monad introduced in an exercise of the previous section. The resulting monad is equivalent to the one defined using `newtype` that we have used up to now.

[a] Chapter 35.2.1 on page 212

If the combined monads `StateT s m` are to be used as state monads, we will certainly want the all-important `get` and `put` operations. Here, we will show definitions in the style of the `mtl package. In addition to the monad transformers themselves,`mtl provides type classes for the essential operations of common monads. For instance, the `MonadState` class, found in Control.Monad.State[7], has `get` and `put` as methods:

```
instance (Monad m) => MonadState s (StateT s m) where
  get   = StateT $ \s -> return (s,s)
  put s = StateT $ \_ -> return ((),s)
```

Note:
`instance (Monad m) => MonadState s (StateT s m)` should be read as: ”For any type `s` and any instance of `Monad` `m`, `s` and `StateT s m` together form an instance of `MonadState`”. `s` and `m` correspond to the state and the inner monad, respectively. `s` is an independent part of the instance specification so that the methods can refer to it – for instance, the type of `put` is `s -> StateT s m ()`.

There are `MonadState` instances for state monads wrapped by other transformers, such as `MonadState s m => MonadState s (MaybeT m)`. They bring us extra convenience by making it unnecessary to lift uses of `get` and `put` explicitly, as the `MonadState` instance for the combined monads handles the lifting for us.

It can also be useful to lift instances that might be available for the inner monad to the combined monad. For instance, all combined monads in which `StateT` is used with an instance of `MonadPlus` can be made instances of `MonadPlus`:

```
instance (MonadPlus m) => MonadPlus (StateT s m) where
  mzero = StateT $ \_ -> mzero
  (StateT x1) `mplus` (StateT x2) = StateT $ \s -> (x1 s) `mplus` (x2 s)
```

The implementations of `mzero` and `mplus` do the obvious thing; that is, delegating the actual work to the instance of the inner monad.

Lest we forget, the monad transformer must have a `MonadTrans`, so that we can use `lift`:

```
instance MonadTrans (StateT s) where
  lift c = StateT $ \s -> c >>= (\x -> return (x,s))
```

[7] http://hackage.haskell.org/packages/archive/mtl/latest/doc/html/Control-Monad-State.html

The `lift` function creates a `StateT` state transformation function that binds the computation in the inner monad to a function that packages the result with the input state. If, for instance, we apply StateT to the List monad, a function that returns a list (i.e., a computation in the List monad) can be lifted into `StateT s []` where it becomes a function that returns a `StateT (s -> [(a,s)])`. I.e. the lifted computation produces *multiple* (value,state) pairs from its input state. This "forks" the computation in StateT, creating a different branch of the computation for each value in the list returned by the lifted function. Of course, applying `StateT` to a different monad will produce different semantics for the `lift` function.

Exercises:

1. Implement `state :: MonadState s m => (s -> (a, s)) -> m a` in terms of `get` and `put`.
2. Are `MaybeT (State s)` and `StateT s Maybe` equivalent? (Hint: one approach is comparing what the `run...T` unwrappers produce in each case.)

37.6 Acknowledgements

This module uses a number of excerpts from All About Monads[8], with permission from its author Jeff Newbern.

ru:Haskell/Monad transformers[9]

[8] http://www.haskell.org/haskellwiki/All_About_Monads

[9] https://ru.wikibooks.org/wiki/Haskell%2FMonad%20transformers

38 Advanced Haskell

39 Monoids

In earlier parts of the book, we have made a few passing allusions to monoids and the `Monoid` type class (most notably when discussing MonadPlus[1]). Here we'll give them a more detailed look and show what makes them useful.

39.1 What is a monoid?

The operation of adding numbers has a handful of properties which are so elementary we don't even think about them when summing numbers up. One of them is *associativity*: when adding three or more numbers it doesn't matter how we group the terms.

```
GHCi> (5 + 6) + 10
21
GHCi> 5 + (6 + 10)
21
```

Another one is that it has an *identity element*, which can be added to any other number without changing its value. That element is the number zero:

```
GHCi> 255 + 0
255
GHCi> 0 + 255
255
```

Addition is not the only binary operation which is associative and has an identity element. Multiplication does too, albeit with a different identity.

```
GHCi> (5 * 6) * 10
300
GHCi> 5 * (6 * 10)
300
GHCi> 255 * 1
255
GHCi> 1 * 255
255
```

We needn't restrict ourselves to arithmetic either. `(++)`, the appending operation for Haskell lists, is another example. It has the empty list as its identity element.

```
GHCi> ([1,2,3] ++ [4,5,6]) ++ [7,8,9]
[1,2,3,4,5,6,7,8,9]
GHCi> [1,2,3] ++ ([4,5,6] ++ [7,8,9])
[1,2,3,4,5,6,7,8,9]
GHCi> [1,2,3] ++ []
[1,2,3]
```

[1] https://en.wikibooks.org/wiki/Haskell%2FMonadPlus%23Relationship%20with%20monoids

```
GHCi> [] ++ [1,2,3]
[1,2,3]
```

It turns out there are a great many associative binary operations with an identity. All of them, by definition, give us examples of *monoids*. We say, for instance, that the integer numbers form a monoid under addition with `0` as identity element.

39.2 The `Monoid` class

Monoids show up very often in Haskell, and so it is not surprising to find there is a type class for them in the core libraries. Here it is:

```
class Monoid a where
    mempty  :: a
    mappend :: a -> a -> a

    mconcat :: [a] -> a
    mconcat = foldr mappend mempty
```

The `mappend` method is the binary operation, and `mempty` is its identity. The third method, `mconcat`, is provided as a bonus; it runs down a list and `mappend`s its elements together in order.

”mappend” is a somewhat long and unwieldy name for a binary function so general, even more so for one which is often used infix. Fortunately, Data.Monoid[2] `Data.Monoid` provides `(<>)`, a convenient operator synonym for `mappend`. In what follows, we will use `mappend` and `(<>)` interchangeably.

As an example, this is the monoid instance for lists:

```
instance Monoid [a] where
    mempty  = []
    mappend = (++)
```

Note that, in this case, `mconcat = foldr (++) []` is equivalent to `concat`, which explains the name of the method.

It is legitimate to think of monoids as types which support *appending* in some sense, though a dose of poetic licence is required. The `Monoid` definition is extremely general and not at all limited to data structures, so ”appending” will be just a metaphor at times.

As we suggested earlier on, numbers (i.e. instances of `Num`) form monoids under both addition and multiplication. That leads to the awkward question of which one to choose when writing the instance. In situations like this one, in which there is no good reason to choose one possibility over the other, the dilemma is averted by creating one `newtype` for each instance:

```
-- | Monoid under addition.
newtype Sum a = Sum { getSum :: a }

-- | Monoid under multiplication.
```

[2] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Monoid.html

```
newtype Product a = Product { getProduct :: a }

instance Num a => Monoid (Sum a) where
    mempty = Sum 0
    Sum x `mappend` Sum y = Sum (x + y)

instance Num a => Monoid (Product a) where
    mempty = Product 1
    Product x `mappend` Product y = Product (x * y)
```

Here is a quick demonstration of `Sum` and `Product`:

```
GHCi> import Data.Monoid
GHCi> Sum 5 <> Sum 6 <> Sum 10
Sum {getSum = 21}
GHCi> mconcat [Sum 5, Sum 6, Sum 10]
Sum {getSum = 21}
GHCi> getSum . mconcat . fmap Sum $ [5, 6, 10]
21
GHCi> getProduct . mconcat . fmap Product $ [5, 6, 10]
300
```

39.2.1 `Monoid` laws

The laws which all instances of `Monoid` must follow simply state the properties we already know: `mappend` is associative and `mempty` is its identity element.

```
(x <> y) <> z = x <> (y <> z) -- associativity
mempty <> x = x               -- left identity
x <> mempty = x               -- right identity
```

Exercises:

1. There are several possible monoid instances for `Bool`. Write at least two of them using `newtype`s, as in the `Sum` and `Product` examples. Be sure to verify the monoid laws hold for your instances [a].

[a] You will later find that two of those instances are defined in `Data.Monoid` already.

39.3 Uses

Which advantages are there in having a class with a pompous name for such a simple concept? As usual in such cases, the key gains are in two associated dimensions: recognisability and generality. Whenever, for instance, you see `(<>)` being used you know that, however the specific instance was defined, the operation being done is associative and has an identity element. Moreover, you also know that if there is an instance of `Monoid` for a type you can take advantage of functions written to deal with monoids in general. As a toy example of such a function, we might take this function that concatenates three lists..

```
threeConcat :: [a] -> [a] -> [a] -> [a]
threeConcat a b c = a ++ b ++ c
```

... and replace all `(++)` with `(<>)`...

```
mthreeConcat :: Monoid m => m -> m -> m -> m
mthreeConcat a b c = a <> b <> c
```

... thus making it work with any `Monoid`. When used on other types the generalised function will behave in an analogous way to the original one, as specified by the monoid laws.

```
GHCi> mthreeConcat "Hello" " " "world!"
"Hello world!"
GHCi> mthreeConcat (Sum 5) (Sum 6) (Sum 10)
Sum {getSum = 21}
```

Monoids are extremely common, and have many interesting practical applications.

The `Writer` monad

A computation of type `Writer w a` returns a value of type `a` while producing extra output of type `w`. A typical use case would be logging, in which each computation produces a log entry for later inspection. The `w` type must be an instance of `Monoid`, and the bind operator of the monad uses `mappend` to accumulate the extra output. In the logging use case, that would mean all entries generated during a series of computations are automatically combined into a single log output.

The `Foldable` class

Monoids play an important role in generalising list-like folding to other data structures. We will study that in detail in the upcoming chapter about the `Foldable` class[3].

Finger trees

Moving on from operations on data structures to data structure implementations, monoids can be used to implement *finger trees*, an efficient and versatile data structure. Its implementation makes use of monoidal values as tags for the tree nodes; and different data structures (such as sequences, priority queues, and search trees) can be obtained simply by changing the involved `Monoid` instance.[4]

Options and settings

In a wholly different context, monoids can be a handy way of treating application options and settings. Two examples are Cabal, the Haskell packaging system (”Package databases are monoids. Configuration files are monoids. Command line flags and sets of command line flags are monoids. Package build information is a monoid.”) and XMonad[5], a tiling window manager implemented in Haskell (”xmonad configuration hooks are monoidal.”) [6]. Below are snippets from a XMonad configuration file (which is just a Haskell program) showing the monoidal hooks in action [7].

[3] Chapter 41 on page 261

[4] This blog post ^{http://apfelmus.nfshost.com/articles/monoid-fingertree.html} , based on a paper by Ralf Hinze and Ross Patterson ^{http://www.soi.city.ac.uk/~ross/papers/FingerTree.html} , contains a brief and accessible explanation on how monoids are used in finger trees.

[5] http://xmonad.org

[6] Sources of the quotes (Haskell Cafe mailing list): http://www.haskell.org/pipermail/haskell-cafe/2009-January/053602.html, http://www.haskell.org/pipermail/haskell-cafe/2009-January/053603.html.

[7] The snippets were taken from Ivy Foster’s example config in the HaskellWiki ^{https://wiki.haskell.org/Xmonad/Config_archive/ivy-foster-xmonad.hs} and XMonad’s XMonad.ManageHook ^{http:

```
-- A ManageHook is a rule, or a combination of rules, for
-- automatically handling specific kinds of windows. It
-- is applied on window creation.

myManageHook :: ManageHook
myManageHook = composeAll
    [ manageConkeror
    , manageDocs
    , manageEmacs
    , manageGimp
    , manageImages
    , manageTerm
    , manageTransient
    , manageVideo
    , manageWeb
    , myNSManageHook scratchpads
    ]

-- manageEmacs, for instance, makes a duplicate of an Emacs
-- window in workspace 3 and sets its opacity to 90%. It
-- looks like this:

-- liftX lifts a normal X action into a Query (as expected by -->)
-- idHook ensures the proper return type
manageEmacs :: ManageHook
manageEmacs =
    className =? "Emacs"
    --> (ask >>= doF . \w -> (copyWindow w "3:emacs"))
    <+> (ask >>= \w -> liftX (setOpacity w 0.9) >> idHook)

-- The hooks are used as fields of the XMonad configuration,
-- which is passed to the IO action that starts XMonad.

myConfig xmproc = defaultConfig
                  { -- Among other fields...
                  , manageHook         = myManageHook
                  }

-- idHook, (<+>), composeAll and (-->) are just user-friendly
-- synonyms for monoid operations, defined in the
-- XMonad.ManageHook module thusly:

-- | The identity hook that returns the WindowSet unchanged.
idHook :: Monoid m => m
idHook = mempty

-- | Infix 'mappend'. Compose two 'ManageHook' from right to left.
(<+>) :: Monoid m => m -> m -> m
(<+>) = mappend

-- | Compose the list of 'ManageHook's.
composeAll :: Monoid m => [m] -> m
composeAll = mconcat

-- | @p --> x@.  If @p@ returns 'True', execute the 'ManageHook'.
--
-- > (-->) :: Monoid m => Query Bool -> Query m -> Query m -- a simpler type
(-->) :: (Monad m, Monoid a) => m Bool -> m a -> m a
p --> f = p >>= \b -> if b then f else return mempty
```

//hackage.haskell.org/packages/archive/xmonad/0.11/doc/html/XMonad-ManageHook.html}
module as of version 0.11.

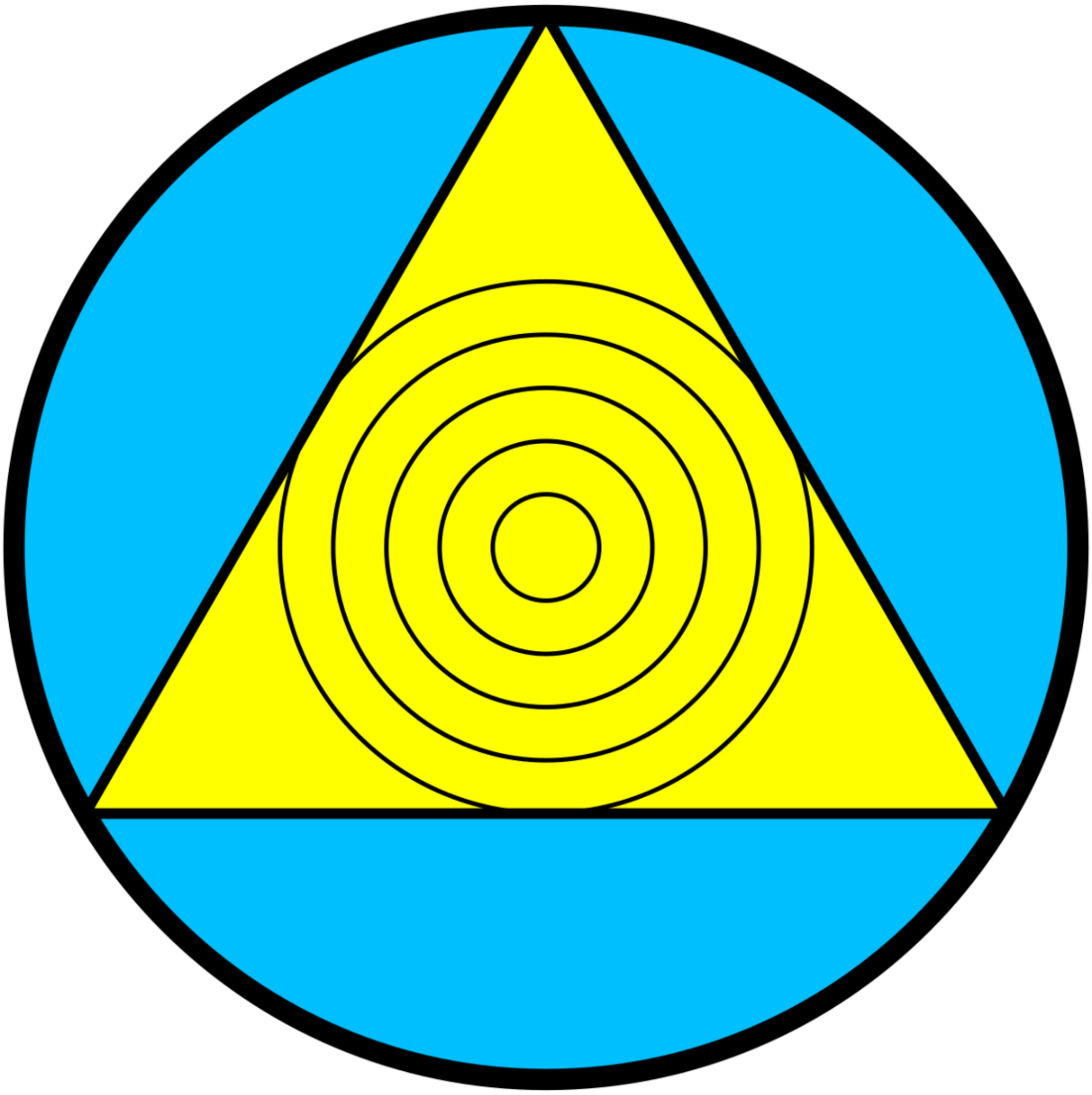


Figure 5 A simple `diagrams` example. The code for it is:

```
mconcat  (fmap
  (circle . (/20)) [1..5])
<> triangle (sqrt 3 / 2)
   lwL 0.01  fc yellow
<> circle 0.5  lwL 0.02
   fc deepskyblue
```

`diagrams`

The `diagrams`[8] package provides a powerful library for generating vectorial images programatically. On a basic level, `(<>)` appears often in code using `diagrams` because squares,

[8] `http://projects.haskell.org/diagrams`

rectangles and other such graphic elements have `Monoid` instances which are used to put them on the top of each other. On a deeper level, most operations with graphic elements are internally defined in terms of monoids, and the implementation takes full advantage of their mathematical properties.

39.4 Homomorphisms

A function `f :: a -> b` between two monoids `a` and `b` is called a **monoid homomorphism** if it preserves the monoid structure, so that:

```
f mempty          = mempty
f (x `mappend` y) = f x `mappend` f y
```

For instance, `length` is an homomorphism between ([],++) and (0,+)

```
length []         = 0
length (xs ++ ys) = length xs + length ys
```

An interesting example "in the wild" of monoids and homomorphisms was identified by Chris Kuklewicz amidst the Google Protocol Buffers API documentation [9] Based on the quotes provided in the referenced comment, we highlight that the property that (in Python)...

```
    MyMessage message;
    message.ParseFromString(str1 + str2);
```

... is equivalent to...

```
    MyMessage message, message2;
    message.ParseFromString(str1);
    message2.ParseFromString(str2);
    message.MergeFrom(message2);
```

... means that `ParseFromString` is a monoid homomorphism. In a hypothetical Haskell implementation, the following equations would hold:

```
parse :: String -> Message
-- these are just equations, not actual code.
parse []         = mempty
parse (xs ++ ys) = parse xs `mergeFrom` parse ys
```

(They wouldn't hold *perfectly*, as parsing might fail, but roughly so.)

Recognising an homomorphism can lead to useful refactorings. For instance, if `mergeFrom` turned out to be an expensive operation it might be advantageous in terms of performance to concatenate the strings before parsing them. `parse` being a monoid homomorphism would then guarantee the same results would be obtained.

[9] Source (Haskell Cafe): `http://www.haskell.org/pipermail/haskell-cafe/2009-January/053709.html`

39.5 Further reading

- Dan Piponi (Sigfpe) on monoids: a blog post overview[10]; a comment about intuition on associativity[11].
- Many monoid related links[12]
- Additional comment on finger trees (Haskell Cafe): FingerTrees[13].
- Additional comments on `Monoid` usage in Cabal (Haskell Cafe): `http://www.haskell.org/pipermail/haskell-cafe/2009-January/053626.html`; `http://www.haskell.org/pipermail/haskell-cafe/2009-January/053721.html`.
- On `diagrams` and monoids: *Monoids: Theme and Variations (Functional Pearl)*[14], by Brent Yorgey.

[10] http://sigfpe.blogspot.com/2009/01/haskell-monoids-and-their-uses.html

[11] http://www.haskell.org/pipermail/haskell-cafe/2009-January/053798.html

[12] http://groups.google.com/group/bahaskell/browse_thread/thread/4cf0164263e0fd6b/42b621f5a4da6019

[13] http://www.haskell.org/pipermail/haskell-cafe/2009-January/053689.html

[14] http://dept.cs.williams.edu/~byorgey/publications.html

40 Applicative functors

When covering the vital `Functor` and `Monad` type classes, we glossed over a third type class: `Applicative`, the class for *applicative functors*. Like monads, applicative functors are functors with extra laws and operations; in fact, `Applicative` is an intermediate class between `Functor` and `Monad`. `Applicative` are a widely used class with a wealth of applications (pardon the pun). It enables the eponymous *applicative style*, a convenient way of structuring functorial computations, and also provides means to express a number of important patterns.

40.1 `Functor` recap

We will begin with a quick review of the Functor class[1] chapter. `Functor` is characterised by the `fmap` function:

```
class Functor f where
    fmap :: (a -> b) -> f a -> f b
```

If a type has an instance of `Functor`, you can use `fmap` to apply a function to values in it. Another way of describing `fmap` is saying that it promotes functions to act on functorial values. To ensure `fmap` works sanely, any instance of `Functor` must comply with the following two laws:

```
fmap id = id                    -- 1st functor law
fmap (g . f) = fmap g . fmap f -- 2nd functor law
```

`Maybe`, for example, has a `Functor` instance, and so we can easily modify the value inside it...

```
Prelude> fmap negate (Just 2)
Just (-2)
```

...as long as it exists, of course.

```
Prelude> fmap negate Nothing
Nothing
```

For extra convenience, `fmap` has an infix synonym, `(<$>)`. It often helps readability, and also suggests how `fmap` can be seen as a different kind of function application.

```
Prelude> negate <$> Just 2
Just (-2)
```

[1] Chapter 27 on page 163

Exercises:

Define instances of `Functor` for the following types:

1. A rose tree, defined as: `data Tree a = Node a [Tree a]`
2. `Either e` for a fixed `e`.
3. The function type `((->) r)`. In this case, `f a` will be `(r -> a)`

40.2 Application in functors

As useful as it is, `fmap` isn't much help if we want to apply a function of two arguments to functorial values. For instance, how could we sum `Just 2` and `Just 3`? The brute force approach would be extracting the values from the `Maybe` wrapper. That, however, would mean having to do tedious checks for `Nothing`. Even worse: in a different `Functor` extracting the value might not even be an option (just think about `IO`).

We could use `fmap` to partially apply `(+)` to the first argument:

```
Prelude> :t (+) <$> Just 2
(+) <$> Just 2 :: Num a => Maybe (a -> a)
```

But now we are stuck: we have a function and a value both wrapped in `Maybe`, and no way of applying one to the other. What we would like to have is an operator with a type akin to `f (a -> b) -> f a -> f b` to apply functions in the context of a functor. If that operator was called `(<*>)`, we would be able to write:

```
(+) <$> Just 2 <*> Just 3
```

Lo and behold - if you try that in GHCi it will just work!

```
Prelude> (+) <$> Just 2 <*> Just 3
Just 5
```

The type of `(<*>)` is:

```
Prelude> :t (<*>)
(<*>) :: Applicative f => f (a -> b) -> f a -> f b
```

`(<*>)` is one of the methods of `Applicative`, the type class of *applicative functors* - functors that support function application within their contexts. Expressions such as `(+) <$> Just 2 <*> Just 3` are said to be written in *applicative style*, which is as close as we can get to regular function application while working with a functor. If you pretend for a moment the `(<$>)`, `(<*>)` and `Just` aren't there, our example looks just like `(+) 2 3`.

40.3 The `Applicative` class

The definition of `Applicative` is:

```
class (Functor f) => Applicative f where
    pure  :: a -> f a
    (<*>) :: f (a -> b) -> f a -> f b
```

Beyond (<*>), the class has a second method, pure, which brings arbitrary values into the functor. As an example, let's have a look at the Maybe instance:

```
instance Applicative Maybe where
    pure                = Just
    (Just f) <*> (Just x) = Just (f x)
    _        <*> _        = Nothing
```

It doesn't do anything surprising: pure wraps the value with Just; (<*>) applies the function to the value if both exist, and results in Nothing otherwise.

40.3.1 Applicative functor laws

Note:
For the lack of a better shorthand, in what follows we will use the jargony word *morphism* to refer to the values to the left of (<*>), which fit the type Applicative f => f (a -> b); that is, the function-like things inserted into an applicative functor. "Morphism" is a term which comes from category theory and which has a much wider meaning, but that needn't concern us now.

Just like Functor, Applicative has a set of laws which reasonable instances should follow. They are:

```
pure id <*> v = v                            -- Identity
pure f <*> pure x = pure (f x)               -- Homomorphism
u <*> pure y = pure ($ y) <*> u              -- Interchange
pure (.) <*> u <*> v <*> w = u <*> (v <*> w) -- Composition
```

Those laws are a bit of a mouthful. They become easier to understand if you think of pure as a way to inject values into the functor in a default, featureless way, so that the result is as close as possible to the plain value. Thus:

- The identity law says that applying the pure id morphism does nothing, exactly like with the plain id function.
- The homomorphism law says that applying a "pure" function to a "pure" value is the same as applying the function to the value in the normal way and then using pure on the result. In a sense, that means pure preserves function application.
- The interchange law says that applying a morphism to a "pure" value pure y is the same as applying pure ($ y) to the morphism. No surprises there - as we have seen in the higher order functions chapter[2], ($ y) is the function that supplies y as argument to another function.
- The composition law says that if (<*>) is used to compose morphisms the composition is associative, like plain function composition [3].

There is also a bonus law about the relation between fmap and (<*>):

```
fmap f x = pure f <*> x                      -- fmap
```

[2] Chapter 19.4.3 on page 122

[3] With plain functions, we have h . g . f = (h . g) . f = h . (g . f). That is why we never bother to use parentheses in the middle of (.) chains.

Applying a ”pure” function with (<*>) is equivalent to using fmap. This law is a consequence of the other ones, so you need not bother with proving it when writing instances of Applicative.

Exercises:

1. Check that the Applicative laws hold for this instance for Maybe
2. Write Applicative instances for
   a. Either e, for a fixed e
   b. ((->) r), for a fixed r

40.4 Déja vu

Does pure remind you of anything?

```
pure :: Applicative f => a -> f a
```

The only difference between that and...

```
return :: Monad m => a -> m a
```

... is the class constraint. pure and return serve the same purpose; that is, bringing values into functors. The uncanny resemblances do not stop here. Back in the chapter about State[4] we mentioned a function called ap...

```
ap :: (Monad m) => m (a -> b) -> m a -> m b
```

... which could be used to make functions with many arguments less painful to handle in monadic code:

```
allTypes :: GeneratorState (Int, Float, Char, Integer, Double, Bool, Int)
allTypes = liftM (,,,,,,) getRandom
                     `ap` getRandom
                     `ap` getRandom
                     `ap` getRandom
                     `ap` getRandom
                     `ap` getRandom
                     `ap` getRandom
```

ap looks a lot like (<*>).

Those, of course, are not coincidences. Monad inherits from Applicative...

```
Prelude> :info Monad
class Applicative m => Monad (m :: * -> *) where
--etc.
```

... because return and (>>=) are enough to implement pure and (<*>) [5].

[4] Chapter 35 on page 207

[5] And if the Monad instance follows the monad laws, the resulting pure and (<*>) will automatically follow the applicative laws.

```
pure = return
(<*>) = ap

ap u v = do
   f <- u
   x <- v
   return (f x)
```

Several other monadic functions have more general applicative versions. Here are a few of them:

Monadic	**Applicative**	**Module** **(where to find the applicative version)**
`(>>)`	`(*>)`	Prelude (GHC 7.10+); Control.Applicative[6]
`liftM2`	`liftA2`	Control.Applicative[7]
`mapM`	`traverse`	Prelude (GHC 7.10+); Data.Traversable[8]
`sequence`	`sequenceA`	Data.Traversable[9]
`forM_`	`for_`	Data.Foldable[10]

Exercises:

1. Write a definition of `(<*>)` using `(>>=)` and `fmap`. Do not use do-notation.
2. Implement
   `liftA5 :: Applicative f => (a -> b -> c -> d -> e -> k)`
   `-> f a -> f b -> f c -> f d -> f e -> f k`

40.5 ZipList

Lists are applicative functors as well. Specialised to lists, the type of `(<*>)` becomes...

```
[a -> b] -> [a] -> [b]
```

... and so `(<*>)` applies a list of functions to another list. But exactly how is that done?

The standard instance of `Applicative` for lists, which follows from the `Monad` instance[11], applies every function to every element, like an explosive version of `map`.

```
Prelude> [(2*),(5*),(9*)] <*> [1,4,7]
[2,8,14,5,20,35,9,36,63]
```

Interestingly, there is another reasonable way of applying a list of functions. Instead of using every combination of functions and values, we can match each function with the value in

[6] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Applicative.html
[7] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Applicative.html
[8] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Traversable.html
[9] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Traversable.html
[10] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Foldable.html
[11] Chapter 32 on page 193

the corresponding position in the other list. A `Prelude` function which can be used for that is `zipWith`:

```
Prelude> :t zipWith
zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]
Prelude> zipWith ($) [(2*),(5*),(9*)] [1,4,7]
[2,20,63]
```

When there are two useful possible instances for a single type, the dilemma is averted by creating a `newtype` which implements one of them. In this case, we have `ZipList`, which lives in Control.Applicative[12]:

```
newtype ZipList a = ZipList { getZipList :: [a] }
```

We have already seen what `<*>` should be for zip-lists; all that is needed is to add the `newtype` wrappers:

```
instance Applicative ZipList where
    (ZipList fs) <*> (ZipList xs) = ZipList (zipWith ($) fs xs)
    pure x                        = undefined -- TODO
```

As for `pure`, it is tempting to use `pure x = ZipList [x]`, following the standard list instance. We can't do that, however, as it violates the applicative laws. According to the identity law:

```
pure id <*> v = v
```

Substituting `(<*>)` and the suggested `pure`, we get:

```
ZipList [id] <*> ZipList xs = ZipList xs
ZipList (zipWith ($) [id] xs) = ZipList xs
```

Now, suppose `xs` is the infinite list `[1..]`:

```
ZipList (zipWith ($) [id] [1..]) = ZipList [1..]
ZipList [1] = ZipList [1..]
[1] = [1..] -- Obviously false!
```

The problem is that `zipWith` produces lists whose length is that of the shortest list passed as argument, and so `(ZipList [id] <*>)` will cut off all elements of the other zip-list after the first. The only way to ensure `zipWith ($) fs` never removes elements is making `fs` infinite. The correct `pure` follows from that:

```
instance Applicative ZipList where
    (ZipList fs) <*> (ZipList xs) = ZipList (zipWith ($) fs xs)
    pure x                        = ZipList (repeat x)
```

The `ZipList` applicative instance offers an alternative to all the zipN and zipWithN functions in Data.List[13] which can be extended to any number of arguments:

```
>>> import Control.Applicative
>>> ZipList [(2*),(5*),(9*)] <*> ZipList [1,4,7]
```

[12] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control.Applicative.html

[13] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-List.html

```
ZipList {getZipList = [2,20,63]}
>>> (,,) <$> ZipList [1,4,9] <*> ZipList [2,8,1] <*> ZipList [0,0,9]
ZipList {getZipList = [(1,2,0),(4,8,0),(9,1,9)]}
>>> liftA3 (,,) (ZipList [1,4,9]) (ZipList [2,8,1]) (ZipList [0,0,9])
ZipList {getZipList = [(1,2,0),(4,8,0),(9,1,9)]}
```

40.6 Sequencing of effects

As we have just seen, the standard `Applicative` instance for lists applies every function in one list to every element of the other. That, however, does not specify `(<*>)` unambiguously. To see why, try to guess what is the result of `[(2*),(3*)]<*>[4,5]` without looking at the example above or the answer just below.

```
Prelude> [(2*),(3*)] <*> [4,5]


--- ...


[8,10,12,15]
```

Unless you were paying very close attention or had already analysed the implementation of `(<*>)`, the odds of getting it right were about even. The other possibility would be `[8,12,10,15]`. The difference is that for the first (and correct) answer the result is obtained by taking the skeleton of the first list and replacing each element by all possible combinations with elements of the second list, while for the other possibility the starting point is the second list.

In more general terms, the difference between is one of *sequencing of effects*. Here, by effects we mean the functorial context, as opposed to the values within the functor (some examples: the skeleton of a list, actions performed in the real world in `IO`, the existence of a value in `Maybe`). The existence of two legal implementations of `(<*>)` for lists which only differ in the sequencing of effects indicates that `[]` is a non-commutative applicative functor. A *commutative* applicative functor, by contrast, leaves no margin for ambiguity in that respect. More formally, a commutative applicative functor is one for which the following holds:

```
liftA2 f u v = liftA2 (flip f) v u -- Commutativity
```

Or, equivalently,

```
f <$> u <*> v = flip f <$> v <*> u
```

By the way, if you hear about *commutative monads* in Haskell, the concept involved is the same, only specialised to `Monad`.

Commutativity (or the lack thereof) affects other functions which are derived from `(<*>)` as well. `(*>)` is a clear example:

```
(*>) :: Applicative f => f a -> f b -> f b
```

`(*>)` combines effects while preserving only the values of its second argument. For monads, it is equivalent to `(>>)`. Here is a demonstration of it using `Maybe`, which is commutative:

```
Prelude> Just 2 *> Just 3
Just 3
Prelude> Just 3 *> Just 2
Just 2
Prelude> Just 2 *> Nothing
Nothing
Prelude> Nothing *> Just 2
Nothing
```

Swapping the arguments does not affect the effects (that is, the being and nothingness of wrapped values). For `IO`, however, swapping the arguments does reorder the effects:

```
Prelude> (print "foo" *> pure 2) *> (print "bar" *> pure 3)
"foo"
"bar"
3
Prelude> (print "bar" *> pure 3) *> (print "foo" *> pure 2)
"bar"
"foo"
2
```

The convention in Haskell is to always implement `(<*>)` and other applicative operators using left-to-right sequencing. Even though this convention helps reducing confusion, it also means appearances sometimes are misleading. For instance, the `(<*)` function is *not* `flip (*>)`, as it sequences effects from left to right just like `(*>)`:

```
Prelude> (print "foo" *> pure 2) <* (print "bar" *> pure 3)
"foo"
"bar"
2
```

For the same reason, `(<**>) :: Applicative f => f a -> f (a -> b) -> f b` from `Control.Applicative` is not `flip (<*>)`. That means it provides a way of inverting the sequencing:

```
>>> [(2*),(3*)] <*> [4,5]
[8,10,12,15]
>>> [4,5] <**> [(2*),(3*)]
[8,12,10,15]
```

An alternative is the Control.Applicative.Backwards[14] module from `transformers`, which offers a `newtype` for flipping the order of effects:

```
newtype Backwards f a = Backwards { forwards :: f a }
```

```
>>> Backwards [(2*),(3*)] <*> Backwards [4,5]
Backwards [8,12,10,15]
```

[14] http://hackage.haskell.org/packages/archive/transformers/latest/doc/html/Control-Applicative-Backwards.

Exercises:

1. For the list functor, implement from scratch (that is, without using anything from `Applicative` or `Monad` directly) both `(<*>)` and its version with the ”wrong” sequencing of effects,
   `(<|*|>) :: Applicative f => f (a -> b) -> f a -> f b`
2. Rewrite the definition of commutativity for a `Monad` using do-notation instead of `ap` or `liftM2`.
3. Are the following applicative functors commutative?
   a. `ZipList`
   b. `((->) r)`
   c. `State s` (Use the `newtype` definition from the `State` chapter[a]. Hint: You may find the answer to exercise 2 of this block useful.)
4. What is the result of `[2,7,8] *> [3,9]`? (Try to guess without writing.)
5. Implement `(<**>)` in terms of other `Applicative` functions.
6. As we have just seen, some functors allow two legal implementations of `(<*>)` which are only different in the sequencing of effects. Why there is not an analogous issue involving `(>>=)`?

[a] Chapter 35.2.1 on page 212

40.7 A sliding scale of power

`Functor`, `Applicative`, `Monad`. Three closely related functor type classes; three of the most important classes in Haskell. Though we have seen many examples of `Functor` and `Monad` in use, and a few of `Applicative`, we have not compared them head to head yet. If we ignore `pure`/`return` for a moment, the characteristic methods of the three classes are:

```
fmap :: Functor f => (a -> b) -> f a -> f b
(<*>) :: Applicative f => f (a -> b) -> f a -> f b
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

While those look like disparate types, we can change the picture with a few aesthetic adjustments. Let’s replace `fmap` by its infix synonym, `(<$>)`; `(>>=)` by its flipped version, `(=<<)`; and tidy up the signatures a bit:

```
(<$>) :: Functor t     =>   (a -> b) -> (t a -> t b)
(<*>) :: Applicative t => t (a -> b) -> (t a -> t b)
(=<<) :: Monad t       => (a -> t b) -> (t a -> t b)
```

Suddenly, the similarities are striking. `fmap`, `(<*>)` and `(=<<)` are all mapping functions over `Functors` [15]. The differences between them are in what is being mapped over in each case:

- `fmap` maps arbitrary functions over functors.

[15] It is not just a question of type signatures resembling each other: the similarity has theoretical ballast. One aspect of the connection is that it is no coincidence that all three type classes have identity and composition laws.

- `(<*>)` maps `t (a -> b)` morphisms[16] over (applicative) functors.
- `(=<<)` maps `a -> t b` functions over (monadic) functors.

The day-to-day differences in uses of `Functor`, `Applicative` and `Monad` follow from what the types of those three mapping functions allow you to do. As you move from `fmap` to `(<*>)` and then to `(>>=)`, you gain in power, versatility and control, at the cost of guarantees about the results. We will now slide along this scale. While doing so, we will use the contrasting terms *values* and *context* to refer to plain values within a functor and to the whatever surrounds them, respectively.

The type of `fmap` ensures that it is impossible to use it to change the context, no matter which function it is given. In `(a -> b) -> t a -> t b`, the `(a -> b)` function has nothing to do with the `t` context of the `t a` functorial value, and so applying it cannot affect the context. For that reason, if you do `fmap f xs` on some list `xs` the number of elements of the list will never change.

```
Prelude> fmap (2*) [2,5,6]
[4,10,12]
```

That can be taken as a safety guarantee or as an unfortunate restriction, depending on what you intend. In any case, `(<*>)` is clearly able to change the context:

```
Prelude> [(2*),(3*)] <*> [2,5,6]
[4,10,12,6,15,18]
```

The `t (a -> b)` morphism carries a context of its own, which is combined with that of the `t a` functorial value. `(<*>)`, however, is subject to a more subtle restriction. While `t (a -> b)` morphisms carry context, within them there are plain `(a -> b)`, which are still unable to modify the context. That means the changes to the context `(<*>)` performs are fully determined by the context of its arguments, and the values have no influence over the resulting context.

```
Prelude> (print "foo" *> pure (2*)) <*> (print "bar" *> pure 3)
"foo"
"bar"
6
Prelude> (print "foo" *> pure 2) *> (print "bar" *> pure 3)
"foo"
"bar"
3
Prelude> (print "foo" *> pure undefined) *> (print "bar" *> pure 3)
"foo"
"bar"
3
```

Thus with list `(<*>)` you know that the length of the resulting list will be the product of the lengths of the original lists, with `IO (<*>)` you know that all real world effect will happen as long as the evaluation terminates, and so forth.

With `Monad`, however, we are in a very different game. `(>>=)` takes a `a -> t b` function, and so it is able to create context from values. That means a lot of flexibility:

```
Prelude> [1,2,5] >>= \x -> replicate x x
```

[16] Chapter 40.3.1 on page 251

```
[1,2,2,5,5,5,5,5]
Prelude> [0,0,0] >>= \x -> replicate x x
[]
Prelude> return 3 >>= \x -> print $ if x < 10 then "Too small" else "OK"
"Too small"
Prelude> return 42 >>= \x -> print $ if x < 10 then "Too small" else "OK"
"OK"
```

Taking advantage of the extra flexibility, however, might mean having less guarantees about, for instance, whether your functions are able to unexpectedly erase parts of a data structure for pathological inputs, or whether the control flow in your application remains intelligible. In some situations there might be performance implications as well, as the complex data dependencies monadic code makes possible might prevent useful refactorings and optimisations. All in all, it is a good idea to only use as much power as needed for the task at hand. If you do need the extra capabilities of `Monad`, go right ahead; however, it is often worth it to check whether `Applicative` or `Functor` are sufficient.

Exercises:
The next few exercises concern the following tree data structure:

```
data AT a = L a | B (AT a) (AT a)
```

1. Write `Functor`, `Applicative` and `Monad` instances for `AT`. Do not use shortcuts such as `pure = return`. The `Applicative` and `Monad` instances should match; in particular, `(<*>)` should be equivalent to `ap`, which follows from the `Monad` instance.
2. Implement the following functions, using either the `Applicative` instance, the `Monad` one or neither of them, if neither is enough to provide a solution. Between `Applicative` and `Monad`, choose the *least* powerful one which is still good enough for the task. Justify your choice for each case in a few words.
   a. `fructify :: AT a -> AT a`, which grows the tree by replacing each leaf `L` with a branch `B` containing two copies of the leaf.
   b. `prune :: a -> (a -> Bool) -> AT a -> AT a`, with `prune z p t` replacing a branch of `t` with a leaf carrying the default value `z` whenever any of the leaves directly on it satisfies the test `p`.
   c. `reproduce :: (a -> b) -> (a -> b) -> AT a -> AT b`, with `reproduce f g t` resulting in a new tree with two modified copies of `t` on the root branch. The left copy is obtained by applying `f` to the values in `t`, and the same goes for `g` and the right copy.
3. There is another legal instance of `Applicative` for `AT` (the reversed sequencing version of the original one doesn't count). Write it. Hint: this other instance can be used to implement
   `sagittalMap :: (a -> b) -> (a -> b) -> AT a -> AT b`
   which, when given a branch, maps one function over the left child tree and the other over the right child tree. (In case you are wondering, ”AT” stands for ”apple tree”. Botanist readers, please forgive the weak metaphors.)

40.8 The monoidal presentation

Back in Understanding monads[17], we saw how the Monad class can be specified using either (>=>) or join instead of (>>=). In a similar way, Applicative also has an alternative presentation, which might be implemented through the following type class:

```
class Functor f => Monoidal f where
    unit  :: f ()
    (*&*) :: f a -> f b -> f (a,b)
```

There are deep theoretical reasons behind the name "monoidal" [18]. In any case, we can informally say that it does look a lot like a monoid: unit provides a default functorial value whose context wraps nothing of interest, and (*&*) combines functorial values by pairing values and combining effects. The Monoidal formulation provides a clearer view of how Applicative manipulates functorial contexts. Naturally, unit and (*&*) can be used to define pure and (<*>), and vice-versa.

The Applicative laws are equivalent to the following set of laws, stated in terms of Monoidal:

```
fmap snd $ unit *&* v = v                     -- Left identity
fmap fst $ u *&* unit = u                     -- Right identity
fmap asl $ u *&* (v *&* w) = (u *&* v) *&* w -- Associativity
-- asl (x, (y, z)) = ((x, y), z)
```

The functions to the left of the ($) are just boilerplate to convert between equivalent types, such as b and ((), b). If you ignore them, the laws are a lot less opaque than in the usual Applicative formulation. By the way, just like for Applicative there is a bonus law, which is guaranteed to hold in Haskell:

```
fmap (g *** h) (u *&* v) = fmap g u *&* fmap h v -- Naturality
-- g *** h = \(x, y) -> (g x, h y)
```

Exercises:

1. Write implementations for unit and (*&*) in terms of pure and (<*>), and vice-versa.
2. Formulate the law of commutative applicative functors (see the Sequencing of effects[a] section) in terms of the Monoidal methods.
3. Write from scratch Monoidal instances for:
   a. ZipList
   b. ((->) r)

[a] Chapter 40.6 on page 255

[17] Chapter 30.3.2 on page 185

[18] For extra details, follow the leads from the corresponding section of the Typeclasseopedia ^{https://wiki.haskell.org/Typeclassopedia#Alternative_formulation} and the blog post by Edward Z. Yang which inspired it ^{http://blog.ezyang.com/2012/08/applicative-functors/} .

41 Foldable

The Foldable type class provides a generalisation of list folding (foldr and friends) and operations derived from it to arbitrary data structures. Besides being extremely useful, Foldable is a great example of how monoids can help formulating good abstractions.

41.1 Deconstructing foldr

foldr is quite a busy function – two binary functions on each side of the first function arrow, with types which use two variables each.

```
foldr :: (a -> b -> b) -> b -> [a] -> b
```

If we are going to generalise foldr, it would be convenient to have something simpler to work with, or at least to be able to break it down into simpler components. What could those components be?

A rough description of list folding would be that it consists of running through the list elements and combining them with a binary function. We happen to know one type class which is all about combining pairs of values: Monoid. If we take foldr f z ...

```
a `f` (b `f` (c `f` z)) -- foldr f z [a,b,c]
```

... and make f = (<>) and z = mempty ...

```
a <> (b <> (c <> mempty)) -- foldr (<>) mempty [a,b,c]
```

... we get mconcat = foldr mappend mempty, which is a simpler, specialised foldr in which we do not need to specify the combining function nor initial accumulator, as we simply use mappend (i.e. (<>)) and mempty:

```
mconcat :: Monoid m => [m] -> m
```

mconcat captures the combine-all-elements aspect of foldr well enough, and covers a few of its use cases:

```
GHCi> mconcat ["Tree", "fingers"] -- concat
"Treefingers"
```

Neat – but surely we don't want to be restricted to folding with Monoid instances only. One way to improve the situation a bit is by realising we can use mconcat to fold a list with elements of any type, as long as we have a function to convert them to some Monoid type:

```
foldMap :: Monoid m => (a -> m) -> [a] -> m
foldMap g = mconcat . fmap g
```

That makes things more interesting already:

```
GHCi> foldMap Sum [1..10]
Sum {getSum = 55}
```

So far so good, but it seems that we are still unable to fold with arbitrary combining functions. It turns out, however, that *any* binary function that fits the `foldr` signature can be used to convert values to a `Monoid` type! The trick is looking at the combining function passed to `foldr` as a function of one argument...

```
foldr :: (a -> (b -> b)) -> b -> [a] -> b
```

... and taking advantage of the fact that `b -> b` functions form a monoid under composition, with `(.)` as `mappend` and `id` as `mempty` [1]. The corresponding `Monoid` instance is available through the `Endo` wrapper from `Data.Monoid` [2]:

```
newtype Endo b = Endo { appEndo :: b -> b }

instance Monoid Endo where
    mempty                = Endo id
    Endo g `mappend` Endo f = Endo (g . f)
```

We can now define...

```
foldComposing :: (a -> (b -> b)) -> [a] -> Endo b
foldComposing f = foldMap (Endo . f)
```

... which makes a `b -> b` function out of each element and composes them all:

```
Endo (f a) <> (Endo (f b) <> (Endo (f c) <> (Endo id))) -- foldComposing f
 [a,b,c]
Endo (f a . (f b . (f c . id)))
-- (<>) and (.) are associative, so we don't actually need the parentheses.

-- As an example, here is a step-by-step evaluation:
foldComposing (+) [1, 2, 3]
foldMap (Endo . (+)) [1, 2, 3]
mconcat (fmap (Endo . (+)) [1, 2, 3])
mconcat (fmap Endo [(+1), (+2), (+3)])
mconcat [Endo (+1), Endo (+2), Endo (+3)]
Endo ((+1) . (+2) . (+3))
Endo (+6)
```

If we apply that function to some `b` value...

```
foldr :: (a -> (b -> b)) -> b -> [a] -> b
foldr f z xs = appEndo (foldComposing f xs) z
```

...we finally recover `foldr`. That means we can define `foldr` in terms of `foldMap`, a function which is much simpler and therefore easier to reason about. For that reason, `foldMap` is the conceptual heart of `Foldable`, the class which generalises `foldr` to arbitrary data structures.

[1] This trick will probably ring familiar if you did the exercise about `foldl` at the end of Higher order functions ^{Chapter19.4.5 on page 124}.

[2] ”Endo” is shorthand for ”endomorphism”, a jargony word for functions from one type to the same type.

Exercises:

1. Write two implementations of foldMap for lists: one in terms of foldr and the other using recursion explicitly.

41.2 The Foldable class

Implementing Foldable for a data structure requires writing just one function: either foldMap or foldr. Foldable, however, has a lot of other methods:

```
-- Abridged definition, with just the method signatures.
class Foldable t where
    foldMap :: Monoid m => (a -> m) -> t a -> m
    foldr :: (a -> b -> b) -> b -> t a -> b

    -- All of the following have default implementations:
    fold :: Monoid m => t m -> m -- generalised mconcat
    foldr' :: (a -> b -> b) -> b -> t a -> b
    foldl :: (b -> a -> b) -> b -> t a -> b
    foldl' :: (b -> a -> b) -> b -> t a -> b
    foldr1 :: (a -> a -> a) -> t a -> a
    foldl1 :: (a -> a -> a) -> t a -> a
    toList :: t a -> [a]
    null :: t a -> Bool
    length :: t a -> Int
    elem :: Eq a => a -> t a -> Bool
    maximum :: Ord a => t a -> a
    minimum :: Ord a => t a -> a
    sum :: Num a => t a -> a
    product :: Num a => t a -> a
```

The extra methods are there so that more efficient implementations can be written if necessary – for instance, if you are writing a highly optimised data structure you don't want foldl' to actually do all the fancy trickery needed to turn a right fold into a left fold. In any case, writing just foldMap or foldr gives you all of the very useful functions listed above for free. And it gets even better: Data.Foldable[3] provides still more functions generalised to any Foldable, including, remarkably, mapM_/traverse_.

Here is a quick demonstration of Foldable using Data.Map[4] [5]:

```
GHCi> import qualified Data.Map as M
GHCi> let testMap = M.fromList $ zip [0..]
 ["Yesterday","I","woke","up","sucking","a","lemon"]
GHCi> length testMap
7
GHCi> sum . fmap length $ testMap
29
GHCi> elem "lemon" testMap
True
GHCi> foldr1 (\x y -> x ++ (' ' : y)) testMap -- Be careful: foldr1 is partial!
```

[3] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Foldable.html

[4] http://hackage.haskell.org/packages/archive/containers/latest/doc/html/Data-Map.html

[5] For more information on Data.Map and other useful data structure implementations with, see the data structures primer ^{https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FData%20structures%20primer} .

```
"Yesterday I woke up sucking a lemon"
GHCi> import Data.Foldable
GHCi> traverse_ putStrLn testMap
Yesterday
I
woke
up
sucking
a
lemon
```

Beyond providing useful generalisations, `Foldable` and `foldMap` suggest a more declarative way of thinking about folds. For instance, instead of describing `sum` as a function which runs across a list (or tree, or whatever the data structure is) accumulating its elements with `(+)`, we might say that it *queries* each element for its value and *summarises* the results of the queries using the `Sum` monoid. Though the difference may seem small, the monoidal summary perspective can help clarifying problems involving folds by separating the core issue of what sort of result we wish to obtain from the details of the data structure being folded.

Exercises:

1. Let's play Spot The Monoid! Here are the rules:For each function, suggest a combination of `mempty`, `mappend` and, if necessary, a function to prepare the values that would allow it to be implemented with `fold` or `foldMap`. No need to bother with `newtype` instances (unless you want to test your solutions with `foldMap`, of course) – for example, "`mempty` is `0` and `mappend` is `(+)`" would be a perfectly acceptable answer for `sum`. If necessary, you can partially apply the functions and use the supplied arguments in the answers. Do not answer every question with `id` and `(.)` - that would be cheating!(Hint: if you need suggestions, have a look at the `Monoid` instances in Data.Monoid[a].)
   a) `product :: (Foldable t, Num a) => t a -> a`
   b) `concat :: Foldable t => t [a] -> [a]`
   c) `concatMap :: Foldable t => (a -> [b]) -> t a -> [b]`
   d) `all :: Foldable t => (a -> Bool) -> t a -> Bool`
   e) `elem :: Eq a => a -> t a -> Bool`
   f) `length :: t a -> Int`
   g) ```
      traverse_ :: (Foldable t, Applicative f) =>
                   (a -> f b) -> t a -> f ()
      ```
   h) ```
      mapM_ :: (Foldable t, Monad m) =>
               (a -> m b) -> t a -> m ()
      ```
   i) `safeMaximum :: Ord a => t a -> Maybe a`
      (like `maximum`, but handling emptiness.)
   j) `find :: Foldable t => (a -> Bool) -> t a -> Maybe a`
   k) ```
      composeL :: Foldable t =>
                  (b -> a -> b) -> t a -> b -> b
      ```
      (equivalent to `foldl`.)

[a] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Monoid.html
      ```

41.3 List-like folding

Foldable includes the toList :: Foldable t => t a -> [a] method. That means any Foldable data structure can be turned into a list; moreover, folding the resulting list will produce the same results than folding the original structure directly. A possible toList implementation in terms of foldMap would be [6]:

```
toList = foldMap (\x -> [x])
```

toList reflects the fact that lists are the **free monoid** for Haskell types. ”Free” here means any value can be promoted to the monoid in a way which neither adds nor erases any information (we can convert values of type a to [a] lists with a single element and back through (\x->[x]) and head in a lossless way) [7].

A related key trait of Foldable is made obvious by toList. Since toList = id for lists, if you are given a function defined as...

```
-- Given a list xs :: [a]
xsAsFoldMap :: Monoid m => (a -> m) -> m
xsAsFoldMap = \f -> foldMap f xs
```

... it is always possible to recover the original list xs by supplying (\x->[x]) to xsAsFoldMap. In this sense, *lists are equivalent to their right folds.* That implies folding with the Foldable operations will unavoidably be a lossy operation if the data structure is more complex than a list. Putting it in another way, we can say that the list-like folds offered by Foldable are less general than folds of the sort we have seen back in Other data structures[8] (formally known as *catamorphisms*), which do make it possible to reconstruct the original structure.

Exercises:

1. This exercise concerns the tree type we used in Other data structures[a]: data Tree a = Leaf a | Branch (Tree a) (Tree a)
   a) Write a Foldable instance for Tree.
   b) Implement treeDepth :: Tree a -> Int, which gives the number of branches from the root of the tree to the furthest leaf. Use either the Foldable or the treeFold catamorphism defined in Other data structures[b]. Are both suggestions actually possible?

[a] Chapter 25.1 on page 145
[b] Chapter 25.1.2 on page 147

[6] Data.Foldable uses a different default implementation for performance reasons.

[7] There is one caveat relating to non-termination with regards to saying lists form a free monoid. For details, see the *Free Monoids in Haskell* ^{http://comonad.com/reader/2015/free-monoids-in-haskell} post by Dan Doel. (Note that the discussion there is quite advanced. You might not enjoy it much right now if you have just been introduced to Foldable.)

[8] Chapter 25 on page 145

41.4 More facts about `Foldable`

`Foldable` is slightly unusual among Haskell classes which are both principled and general-purpose in that it has no laws of its own. The closest thing is the following property, which strictly speaking is not a law (as it is guaranteed to hold whatever the instance is): given a monoid homomorphism[9] `g`,

```
foldMap (g . f) = g . foldMap f
```

Switching from `foldMap (g . f)` to `g . foldMap f` can be advantageous, as it means applying `g` only to the result of the fold, rather than to the potentially many elements in the structure being folded.

If the `Foldable` structure is a `Functor` as well, it also automatically holds that...

```
foldMap f = fold . fmap f
```

... and thus we get, after applying the second functor law[10] and the property just above:

```
foldMap g . fmap f = foldMap (g . f) = g . foldMap f
```

Though the presence of a method such as `foldMap` might suggest that any `Foldable` types should have `Functor` instances as well, `Functor` is not actually a superclass of `Foldable`. That makes it possible to give `Foldable` instances to structures that, for whatever reason, cannot be `Functor`s. The most common example are the sets[11] from Data.Set[12]. Element types for those sets must be instances of `Ord`, and therefore their `map` function cannot be used as `fmap`, which has no additional class constraints. That, however, does not deny `Data.Set.Set` an useful `Foldable` instance.

```
GHCi> import qualified Data.Set as S
GHCi> let testSet = S.fromList [1,3,2,5,5,0]
GHCi> testSet
fromList [0,1,2,3,5]
GHCi> import Data.Foldable
GHCi> toList testSet
[0,1,2,3,5]
GHCi> foldMap show testSet
"01235"
```

[9] Chapter 39.4 on page 247

[10] Chapter 27.2.1 on page 164

[11] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FData%20structures%20primer%23Variations

[12] http://hackage.haskell.org/packages/archive/containers/latest/doc/html/Data-Set.html

Exercises:

1. a) Write the monoid instance for pairs,

   ```
   (Monoid a, Monoid b) => Monoid (a,b)
   ```

   b) Prove that `fst` and `snd` are monoid homomorphisms.

   c) Use the monoid homomorphism property of `foldMap` presented above to prove that

   ```
   foldMap f &&& foldMap g = foldMap (f &&& g)
   ```

   where

   ```
   f &&& g = \x -> (f x, g x)
   ```

This exercise is based on a message by Edward Kmett [a].

[a] Source (Haskell Café): https://mail.haskell.org/pipermail/haskell-cafe/2015-February/118152.html

42 Traversable

We already have studied four of the five type classes in the Prelude that can be used for data structure manipulation: `Functor`, `Applicative`, `Monad` and `Foldable`. The fifth one is `Traversable` [1]. To traverse means to walk across, and that is exactly what `Traversable` generalises: walking across a structure, collecting results at each stop.

42.1 Functors made for walking

If traversing means walking across, though, we have been performing traversals for a long time already. Consider the following plausible `Functor` and `Foldable` instances for lists:

```
instance Functor [] where
    fmap _ []     = []
    fmap f (x:xs) = f x : fmap f xs

instance Foldable [] where
    foldMap _ []     = mempty
    foldMap f (x:xs) = f x <> foldMap f xs
```

`fmap f` walks across the list, applies `f` to each element and collects the results by rebuilding the list. Similarly, `foldMap f` walks across the list, applies `f` to each element and collects the results by combining them with `mappend`. `Functor` and `Foldable`, however, are not enough to express all useful ways of traversing. For instance, suppose we have the following `Maybe`-encoded test for negative numbers...

```
deleteIfNegative :: (Num a, Ord a) => a -> Maybe a
deleteIfNegative x = if x < 0 then Nothing else Just x
```

... and we want to use it to implement...

```
rejectWithNegatives :: (Num a, Ord a) => [a] -> Maybe [a]
```

... which gives back the original list wrapped in `Just` if there are no negative elements in it, and `Nothing` otherwise. Neither `Foldable` nor `Functor` on their own would help. Using `Foldable` would replace the structure of the original list with that of whatever `Monoid` we pick for folding, and there is no way of twisting that into giving either the original list or `Nothing` [2]. As for `Functor`, `fmap` might be attractive at first...

[1] Strictly speaking, we should refer to the five classes in the GHC Prelude, as `Applicative`, `Foldable` and `Traversable` aren't officially part of the Prelude yet according to the Haskell Report ^{`https://www.haskell.org/onlinereport/haskell2010`} . It is just a matter of time for them to be included, though.

[2] One thing to attempt would be exploiting the `Monoid a => Monoid (Maybe a)` instance from Data.Monoid ^{`http://hackage.haskell.org/packages/archive/base/latest/doc/html/`

```
GHCi> let testList = [-5,3,2,-1,0]
GHCi> fmap deleteIfNegative testList
[Nothing,Just 3,Just 2,Nothing,Just 0]
```

... but then we would need a way to turn a list of Maybe into Maybe a list. If you squint hard enough, that looks somewhat like a fold. Instead, however, of merely combining the values and destroying the list, we need to combine the Maybe contexts of the values and recreate the list structure within the combined context. Fortunately, there is a type class which is essentially about combining Functor contexts: Applicative [3]. Applicative, in turn, leads us to the class we need: Traversable.

```
instance Traversable [] where
    -- sequenceA :: Applicative f => [f a] -> f [a]
    sequenceA []     = pure []
    sequenceA (u:us) = (:) <$> u <*> sequenceA us

-- Or, equivalently:
instance Traversable [] where
    sequenceA us = foldr (\u v -> (:) <$> u <*> v) (pure []) us
```

Traversable is to Applicative contexts what Foldable is to Monoid values. From that point of view, sequenceA is analogous to fold – it creates an applicative summary of the contexts within a structure, and then rebuilds the structure in the new context. sequenceA is the function we were looking for:

```
GHCi> let rejectWithNegatives = sequenceA . fmap deleteIfNegative
GHCi> :t rejectWithNegatives
rejectWithNegatives
  :: (Num a, Ord a, Traversable t) => t a -> Maybe (t a)
GHCi> rejectWithNegatives testList
Nothing
GHCi> rejectWithNegatives [0..10]
Just [0,1,2,3,4,5,6,7,8,9,10]
```

These are the methods of Traversable:

```
class (Functor t, Foldable t) => Traversable t where
    traverse  :: Applicative f => (a -> f b) -> t a -> f (t b)
    sequenceA :: Applicative f => t (f a) -> f (t a)

    -- These methods have default definitions.
    -- They are merely specialised versions of the other two.
    mapM      :: Monad m => (a -> m b) -> t a -> m (t b)
    sequence  :: Monad m => t (m a) -> m (t a)
```

If sequenceA is analogous to fold, traverse is analogous to foldMap. They can be defined in terms of each other, and therefore a minimal implementation of Traversable just needs to supply one of them:

```
traverse f = sequenceA . fmap f
sequenceA = traverse id
```

Data-Monoid.html} . If you try that, however, you will see it can't possibly give the desired results.

[3] The monoidal presentation ^{https://en.wikibooks.org/wiki/Haskell%2FApplicative%20functors%20II%23The%20monoidal%20presentation} of Applicative makes that very clear.

Rewriting the list instance using traverse makes the parallels with Functor and Foldable obvious:

```
instance Traversable [] where
    traverse _ []     = pure []
    traverse f (x:xs) = (:) <$> f x <*> traverse f xs

-- Or, equivalently:
instance Traversable [] where
    traverse f xs = foldr (\x v -> (:) <$> f x <*> v) (pure []) xs
```

In general, it is better to write traverse when implementing Traversable, as the default definition of traverse performs, in principle, two runs across the structure (one for fmap and another for sequenceA).

We can cleanly define rejectWithNegatives directly in terms of traverse:

```
rejectWithNegatives :: (Num a, Ord a, Traversable t) => t a -> Maybe (t a)
rejectWithNegatives = traverse deleteIfNegative
```

Exercises:

1. Give the Tree from Other data structures[a] a Traversable instance. The definition of Tree is:

   ```
   data Tree a = Leaf a | Branch (Tree a) (Tree a)
   ```

[a] Chapter 25.1 on page 145

42.2 Interpretations of Traversable

Traversable structures can be walked over using the applicative functor of your choice. The type of traverse...

```
traverse :: (Applicative f, Traversable t) => (a -> f b) -> t a -> f (t b)
```

... resembles that of mapping functions we have seen in other classes. Rather than using its function argument to insert functorial contexts under the original structure (as might be done with fmap) or to modify the structure itself (as (>>=) does), traverse adds an extra layer of context *on the top* of the structure. Said in another way, traverse allows for *effectful traversals* – traversals which produce an overall effect (i.e. the new outer layer of context).

If the structure below the new layer is recoverable at all, it will match the original structure (the values might have changed, of course). Here is an example involving nested lists:

```
GHCi> traverse (\x -> [0..x]) [0..3]
[[0,0,0,0],[0,0,0,1],[0,0,0,2],[0,0,0,3],[0,0,1,0],[0,0,1,1]
,[0,0,1,2],[0,0,1,3],[0,0,2,0],[0,0,2,1],[0,0,2,2],[0,0,2,3]
,[0,1,0,0],[0,1,0,1],[0,1,0,2],[0,1,0,3],[0,1,1,0],[0,1,1,1]
,[0,1,1,2],[0,1,1,3],[0,1,2,0],[0,1,2,1],[0,1,2,2],[0,1,2,3]
]
```

The inner lists retain the structure the original list – all of them have four elements. The outer list is the new layer, corresponding to the introduction of nondeterminism through allowing each element to vary from zero to its (original) value.

We can also understand `Traversable` by focusing on `sequenceA` and how it *distributes* context.

```
GHCi> sequenceA [[1,2,3,4],[5,6,7]]
[[1,5],[1,6],[1,7],[2,5],[2,6],[2,7]
,[3,5],[3,6],[3,7],[4,5],[4,6],[4,7]
]
```

In this example, `sequenceA` can be seen distributing the old outer structure into the new outer structure, and so the new inner lists have two elements, just like the old outer list. The new outer structure is a list of twelve elements, which is exactly what you would expect from combining with `(<*>)` one list of four elements with another of three elements. One interesting aspect of the distribution perspective is how it helps making sense of why certain functors cannot possibly have instances of `Traversable` (how would one distribute an `IO` action? Or a function?).

Exercises:
Having the applicative functors[a] chapter fresh in memory can help with the following exercises.

1. Consider a representation of matrices[b] as nested lists, with the inner lists being the rows. Use `Traversable` to implement

   ```
   transpose :: [[a]] -> [[a]]
   ```
   which transposes a matrix (i.e. changes columns into rows and vice-versa). For the purposes of this exercise, we don't care about how fake ”matrices” with rows of different sizes are handled.
2. Explain what `traverse mappend` does.
3. Time for a round of Spot The Applicative Functor. Consider:

   ```
   mapAccumL :: Traversable t =>
                (a -> b -> (a, c)) -> a -> t b -> (a, t c)
   ```
   Does its type remind you of anything? Use the appropriate `Applicative` to implement it with `Traversable`. As further guidance, here is the description of `mapAccumL` in the Data.Traversable[c] documentation:

   > The mapAccumL function behaves like a combination of fmap and foldl; it applies a function to each element of a structure, passing an accumulating parameter from left to right, and returning a final value of this accumulator together with the new structure.

[a] Chapter 40 on page 249

[b] https://en.wikipedia.org/wiki/Matrix%20%28mathematics%29

[c] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Traversable.html

42.3 The Traversable laws

Sensible instances of Traversable have a set of laws to follow. There are the following two laws:

```
traverse Identity = Identity -- identity
traverse (Compose . fmap g . f) = Compose . fmap (traverse g) . traverse f --
 composition
```

Plus a bonus law, which is guaranteed to hold:

```
-- If t is an applicative homomorphism, then
t . traverse f = traverse (t . f) -- naturality
```

Those laws are not exactly self-explanatory, so let's have a closer look at them. Starting from the last one: an applicative homomorphism is a function which preserves the Applicative operations, so that:

```
-- Given a choice of f and g, and for any a,
t :: (Applicative f, Applicative g) => f a -> g a

t (pure x) = pure x
t (x <*> y) = t x <*> t y
```

Note that not only this definition is analogous to the one of monoid homomorphisms[4] which we have seen earlier on but also that the naturality law mirrors exactly the property about foldMap and monoid homomorphisms seen in the chapter about Foldable[5].

The identity law involves Identity, the dummy functor:

```
newtype Identity a = Identity { runIdentity :: a }

instance Functor Identity where
    fmap f (Identity x) = Identity (f x)

instance Applicative Identity where
    pure x = Identity x
    Identity f <*> Identity x = Identity (f x)
```

The law says that all traversing with the Identity constructor does is wrap the structure with Identity, which amounts to doing nothing (as the original structure can be trivially recovered with runIdentity). The Identity constructor is thus the identity traversal, which is very reasonable indeed.

The composition law, in turn, is stated in terms of the Compose functor:

```
newtype Compose f g a = Compose { getCompose :: f (g a) }

instance (Functor f, Functor g) => Functor (Compose f g) where
    fmap f (Compose x) = Compose (fmap (fmap f) x)

instance (Applicative f, Applicative g) => Applicative (Compose f g) where
    pure x = Compose (pure (pure x))
    Compose f <*> Compose x = Compose ((<*>) <$> f <*> x)
```

[4] Chapter 39.4 on page 247

[5] Chapter 42 on page 269

Compose performs composition *of functors*. Composing two Functors results in a Functor, and composing two Applicatives results in an Applicative [6]. The instances are the obvious ones, threading the methods one further functorial layer down.

The composition law states that it doesn't matter whether we perform two traversals separately (right side of the equation) or compose them in order to walk across the structure only once (left side). It is analogous, for instance, to the second functor law. The fmaps are needed because the second traversal (or the second part of the traversal, for the left side of the equation) happens below the layer of structure added by the first (part). Compose is needed so that the composed traversal is applied to the correct layer.

Identity and Compose are available from Data.Functor.Identity[7] and Data.Functor.Compose[8] respectively.

The laws can also be formulated in terms of sequenceA:

```
sequenceA . fmap Identity = Identity -- identity
sequenceA . fmap Compose = Compose . fmap sequenceA . sequenceA -- composition
-- For any applicative homomorphism t:
t . sequenceA = sequenceA . fmap t -- naturality
```

Though it's not immediately obvious, several desirable characteristics of traversals follow from the laws, including [9]:

- Traversals do not skip elements.
- Traversals do not visit elements more than once.
- traverse pure = pure
- Traversals cannot modify the original structure (it is either preserved or fully destroyed).

42.4 Recovering fmap and foldMap

We still have not justified the Functor and Foldable class constraints of Traversable. The reason for them is very simple: as long as the Traversable instance follows the laws traverse is enough to implement both fmap and foldMap. For fmap, all we need is to use Identity to make a traversal out of an arbitrary function:

```
fmap f = runIdentity . traverse (Identity . f)
```

To recover foldMap, we need to introduce a third utility functor: Const from Control.Applicative[10]:

```
newtype Const a b = Const { getConst :: a }

instance Functor (Const a) where
    fmap _ (Const x) = Const x
```

[6] Remarkably, however, composing two Monads does not necessarily result in a Monad.

[7] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Functor-Identity.html

[8] http://hackage.haskell.org/packages/archive/transformers/latest/doc/html/Data-Functor-Compose.html

[9] For technical details, check the papers cited by the Data.Traversable ^{http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Traversable.html} documentation.

[10] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control.Applicative.html

Const is a *constant functor*. A value of type Const a b does not actually contain a b value. Rather, it holds an a value which is unaffected by fmap. For our current purposes, the truly interesting instance is the Applicative one

```
instance Monoid a => Applicative (Const a) where
    pure _ = Const mempty
    Const x <*> Const y = Const (x `mappend` y)
```

(<*>) simply combines the values in each context with mappend [11]. We can exploit that to make a traversal out of any Monoid m => a -> m function that we might pass to foldMap. Thanks to the instance above, the traversal then becomes a fold:

```
foldMap f = getConst . traverse (Const . f)
```

We have just recovered from traverse two functions which on the surface appear to be entirely different, and all we had to do was pick two different functors. That is a taste of how powerful an abstraction functors are [12].

[11] This is a great illustration of how Applicative combines contexts monoidally. If we remove the values within the context, the applicative laws in monoidal presentation ^{https://en.wikibooks.org/wiki/Haskell%2FApplicative%20functors%20II%23The%20monoidal%20presentation} match the monoid laws exactly.

[12] A prime example, and one of clear practical relevance at that, is that great ode to functors, the lens ^{https://hackage.haskell.org/package/lens} library.

43 Arrow tutorial

Arrows provide an alternative to the usual way of structuring computations with the basic functor classes. This chapter provides a hands-on tutorial about them, while the next one, Understanding arrows[1], complements it with a conceptual overview. We recommend you to start with the tutorial, so that you get to taste what programming with arrows feel like. You can of course switch back and forth between the tutorial and the first part of *Understanding arrows*if you prefer going at a slower pace. Be sure to follow along every step of the tutorial on GHC(i).

43.1 Stephen's Arrow Tutorial

In this tutorial, I will create my own arrow, show how to use the arrow `proc` notation, and show how `ArrowChoice` works. We will end up with a simple game of Hangman.

First, we give a language pragma to enable the arrow do notation in the compiler:

```
{-# LANGUAGE Arrows #-}
```

And then, some imports:

```
module Main where

import Control.Arrow
import Control.Monad
import qualified Control.Category as Cat
import Data.List
import Data.Maybe
import System.Random
```

Any Haskell function can behave as an arrow, because there is an Arrow instance for the function type constructor `(->)`. In this tutorial I will build a more interesting arrow than this, with the ability to maintain state (something that a plain Haskell function arrow cannot do). Arrows can produce all sorts of effects, including I/O, but we'll just explore some simple examples.

We'll call our new arrow `Circuit` to suggest that we can visualize arrows as circuits.[2]

[1] Chapter 44 on page 287

[2] This interpretation of arrows-as-circuits is loosely based on the Yampa functional reactive programming library.

43.2 Type definition for Circuit

A plain Haskell function treated as an arrow has type a -> b. Our Circuit arrow has two distinguishing features: First, we wrap it in a newtypedeclaration to cleanly define an Arrow instance. Second, in order for the circuit to maintain its own internal state, our arrow returns a replacement for itself along with the normal b output value.

```
newtype Circuit a b = Circuit { unCircuit :: a -> (Circuit a b, b) }
```

To make this an arrow, we need to make it an instance of both Category and Arrow. Throughout these definitions, we always replace each Circuit with the new version of itself that it has returned.

```
instance Cat.Category Circuit where
    id = Circuit $ \a -> (Cat.id, a)
    (.) = dot
      where
        (Circuit cir2) `dot` (Circuit cir1) = Circuit $ \a ->
            let (cir1', b) = cir1 a
                (cir2', c) = cir2 b
            in  (cir2' `dot` cir1', c)
```

The Cat.id function replaces itself with a copy of itself without maintaining any state. The purpose of the (.) function is to chain two arrows together from right to left. (>>>) and (<<<) are based on (.). It needs to replace itself with the ‘dot‘ of the two replacements returned by the execution of the argument Circuits.

```
instance Arrow Circuit where
    arr f = Circuit $ \a -> (arr f, f a)
    first (Circuit cir) = Circuit $ \(b, d) ->
        let (cir', c) = cir b
        in  (first cir', (c, d))
```

arr lifts a plain Haskell function as an arrow. Like with id, the replacement it gives is just itself, since a plain Haskell function can't maintain state.

Now we need a function to run a circuit:

```
runCircuit :: Circuit a b -> [a] -> [b]
runCircuit _   []     = []
runCircuit cir (x:xs) =
    let (cir',x') = unCircuit cir x
    in  x' : runCircuit cir' xs
```

For mapAccumL fans like me, this can alternatively be written as

```
runCircuit :: Circuit a b -> [a] -> [b]
runCircuit cir inputs =
    snd $ mapAccumL (\cir x -> unCircuit cir x) cir inputs
```

or, after eta-reduction, simply as:

```
runCircuit :: Circuit a b -> [a] -> [b]
runCircuit cir = snd . mapAccumL unCircuit cir
```

43.3 Circuit primitives

Let's define a generalized accumulator to be the basis for our later work. `accum'` is a less general version of `accum`.

```
-- | Accumulator that outputs a value determined by the supplied function.
accum :: acc -> (a -> acc -> (b, acc)) -> Circuit a b
accum acc f = Circuit $ \input ->
    let (output, acc') = input `f` acc
    in  (accum acc' f, output)

-- | Accumulator that outputs the accumulator value.
accum' :: b -> (a -> b -> b) -> Circuit a b
accum' acc f = accum acc (\a b -> let b' = a `f` b in (b', b'))
```

Here is a useful concrete accumulator which keeps a running total of all the numbers passed to it as inputs.

```
total :: Num a => Circuit a a
total = accum' 0 (+)
```

We can run this circuit, like this:

```
*Main> runCircuit total [1,0,1,0,0,2]
[1,1,2,2,2,4]
*Main>
```

43.4 Arrow proc notation

Here is a statistical mean function:

```
mean1 :: Fractional a => Circuit a a
mean1 = (total &&& (const 1 ^>> total)) >>> arr (uncurry (/))
```

It maintains two accumulator cells, one for the sum, and one for the number of elements. It splits the input using the "fanout" operator `&&&` and before the input of the second stream, it discards the input value and replaces it with 1.

`const 1 ^>> total` is shorthand for `arr (const 1) >>> total`. The first stream is the sum of the inputs. The second stream is the sum of 1 for each input (i.e. a count of the number of inputs). Then, it merges the two streams with the `(/)` operator.

Here is the same function, but written using arrow `proc` notation:

```
mean2 :: Fractional a => Circuit a a
mean2 = proc value -> do
    t <- total -< value
    n <- total -< 1
    returnA -< t / n
```

The `proc` notation describes the same relationship between the arrows, but in a totally different way. Instead of explicitly describing the wiring, you glue the arrows together using variable bindings and pure Haskell expressions, and the compiler works out all the `arr, (>>>), (&&&)` stuff for you. Arrow `proc` notation also contains a pure 'let' statement exactly like the monadic `do` one.

`proc` is the keyword that introduces arrow notation, and it binds the arrow input to a pattern (`value` in this example). Arrow statements in a `do` block take one of these forms:

- *`variable binding pattern <- arrow -< pure expression giving arrow input`*
- *`arrow -< pure expression giving arrow input`*

Like with monads, the `do` keyword is needed only to combine multiple lines using the variable binding patterns with `<-`. As with monads, the last line isn't allowed to have a variable binding pattern, and the output value of the last line is the output value of the arrow. `returnA` is an arrow just like 'total' is (in fact, `returnA` is just the identity arrow, defined as `arr id`).

Also like with monads, lines other than the last line may have no variable binding, and you get the effect only, discarding the return value. In `Circuit`, there would never be a point in doing this (since no state can escape except through the return value), but in many arrows there would be.

As you can see, for this example the `proc` notation makes the code much more readable. Let's try them:

```
*Main> runCircuit mean1 [0,10,7,8]
[0.0,5.0,5.666666666666667,6.25]
*Main> runCircuit mean2 [0,10,7,8]
[0.0,5.0,5.666666666666667,6.25]
*Main>
```

43.5 Hangman: Pick a word

Now for our Hangman game. Let's pick a word from a dictionary:

```
generator :: Random a => (a, a) -> StdGen -> Circuit () a
generator range rng = accum rng $ \() rng -> randomR range rng

dictionary = ["dog", "cat", "bird"]

pickWord :: StdGen -> Circuit () String
pickWord rng = proc () -> do
    idx <- generator (0, length dictionary-1) rng -< ()
    returnA -< dictionary !! idx
```

With `generator`, we're using the accumulator functionality to hold our random number generator. `pickWord` doesn't introduce anything new, except that the generator arrow is constructed by a Haskell function that takes arguments. Here is the output:

```
*Main> rng <- getStdGen
*Main> runCircuit (pickWord rng) [(), (), ()]
["dog","bird","dog"]
*Main>
```

We will use these little arrows in a minute. The first returns `True` the first time, then `False`forever afterwards:

```
oneShot :: Circuit () Bool
oneShot = accum True $ \_ acc -> (acc, False)
```

```
*Main> runCircuit oneShot [(), (), (), (), ()]
[True,False,False,False,False]
```

The second stores a value and returns it, when it gets a new one:

```
delayedEcho :: a -> Circuit a a
delayedEcho acc = accum acc (\a b -> (b,a))
```

which can be shortened to:

```
delayedEcho :: a -> Circuit a a
delayedEcho acc = accum acc (flip (,))
```

```
*Main> runCircuit (delayedEcho False) [True, False, False, False, True]
[False,True,False,False,False]
```

The game's main arrow will be executed repeatedly, and we would like to pick the word only once on the first iteration, and have it remember it for the rest of the game. Rather than just mask its output on subsequent loops, we'd prefer to actually run `pickWord` only once (since in a real implementation it could be very slow). However, as it stands, the data flow in a Circuit **must** go down **all** the paths of component arrows. In order to allow the data flow to go down one path and not another, we need to make our arrow an instance of `ArrowChoice`. Here's the minimal definition:

```
instance ArrowChoice Circuit where
    left orig@(Circuit cir) = Circuit $ \ebd -> case ebd of
        Left b -> let (cir', c) = cir b
                  in  (left cir', Left c)
        Right d -> (left orig, Right d)

getWord :: StdGen -> Circuit () String
getWord rng = proc () -> do
    -- If this is the first game loop, run pickWord. mPicked becomes Just
 <word>.
    -- On subsequent loops, mPicked is Nothing.
    firstTime <- oneShot -< ()
    mPicked <- if firstTime
        then do
            picked <- pickWord rng -< ()
            returnA -< Just picked
        else returnA -< Nothing
    -- An accumulator that retains the last 'Just' value.
    mWord <- accum' Nothing mplus -< mPicked
    returnA -< fromJust mWord
```

Because `ArrowChoice` is defined, the compiler now allows us to put an `if` after `<-`, and thus choose which arrow to execute (either run `pickWord`, or skip it). Note that this is not a normal Haskell `if`: The compiler implements this using `ArrowChoice`. The compiler also implements `case` here in the same way.

It is important to understand that none of the local name bindings, including the `proc` argument, is in scope between `<-` and `-<` except in the condition of an `if` or `case`. For example, this is illegal:

```
{-
proc rng -> do
    idx <- generator (0, length dictionary-1) rng -< ()  -- ILLEGAL
    returnA -< dictionary !! idx
-}
```

The arrow to execute, here `generator (0, length dictionary -1) rng`, is evaluated in the scope that exists outside the 'proc' statement. `rng` does not exist in this scope. If you think about it, this makes sense, because the arrow is constructed at the beginning only (outside `proc`). If it were constructed for each execution of the arrow, how would it keep its state?

Let's try `getWord`:

```
*Main> rng <- getStdGen
*Main> runCircuit (getWord rng) [(), (), (), (), (), ()]
["dog","dog","dog","dog","dog","dog"]
*Main>
```

43.6 Hangman: Main program

Now here is the game:

```
attempts :: Int
attempts = 5

livesLeft :: Int -> String
livesLeft hung = "Lives: ["
              ++ replicate (attempts - hung) '#'
              ++ replicate hung ' '
              ++ "]"

hangman :: StdGen -> Circuit String (Bool, [String])
hangman rng = proc userInput -> do
    word <- getWord rng -< ()
    let letter = listToMaybe userInput
    guessed <- updateGuess -< (word, letter)
    hung <- updateHung -< (word, letter)
    end <- delayedEcho True -< not (word == guessed || hung >= attempts)
    let result = if word == guessed
                   then [guessed, "You won!"]
                   else if hung >= attempts
                       then [guessed, livesLeft hung, "You died!"]
                       else [guessed, livesLeft hung]
    returnA -< (end, result)
  where
    updateGuess :: Circuit (String, Maybe Char) String
    updateGuess = accum' (repeat '_') $ \(word, letter) guess ->
        case letter of
            Just l  -> map (\(w, g) -> if w == l then w else g) (zip word guess)
            Nothing -> take (length word) guess

    updateHung :: Circuit (String, Maybe Char) Int
    updateHung = proc (word, letter) -> do
        total -< case letter of
            Just l  -> if l `elem` word then 0 else 1
            Nothing -> 0
```

```
main :: IO ()
main = do
    rng <- getStdGen
    interact $ unlines                      -- Concatenate lines out output
        . ("Welcome to Arrow Hangman":)     -- Prepend a greeting to the output
        . concat . map snd . takeWhile fst  -- Take the [String]s as long as the
 first element of the tuples is True
        . runCircuit (hangman rng)          -- Process the input lazily
        . ("":)                             -- Act as if the user pressed ENTER
 once at the start
        . lines                             -- Split input into lines
```

And here's an example session. For best results, compile the game and run it from a terminal rather than from GHCi:

```
Welcome to Arrow Hangman
___
Lives: [#####]
a
___
Lives: [#### ]
g
__g
Lives: [#### ]
d
d_g
Lives: [#### ]
o
dog
You won!
```

43.7 Advanced stuff

In this section I will complete the coverage of arrow notation.

43.7.1 Combining arrow commands with a function

We implemented `mean2` like this:

```
mean2 :: Fractional a => Circuit a a
mean2 = proc value -> do
    t <- total -< value
    n <- total -< 1
    returnA -< t / n
```

GHC defines a banana bracket syntax for combining arrow statements with a function that operates on arrows. (In Ross Paterson's paper [3] a `form` keyword is used, but GHC adopted the banana bracket instead.) Although there's no real reason to, we can write `mean` like this:

```
mean3 :: Fractional a => Circuit a a
mean3 = proc value -> do
```

[3] Ross Paterson's Paper specifying arrow `proc` notation ^{http://www.soi.city.ac.uk/~ross/papers/notation.html}

```
    (t, n) <- (| (&&&) (total -< value) (total -< 1) |)
    returnA -< t / n
```

The first item inside the `(| ... |)` is a function that takes any number of arrows as input and returns an arrow. Infix notation cannot be used here. It is followed by the arguments, which are in the form of proc statements. These statements may contain `do` and bindings with `<-` if you like. Each argument is translated into an arrow and given as an argument to the function `(&&&)`.

You may ask, what is the point of this? We can combine arrows quite happily without the `proc` notation. Well, the point is that you get the convenience of using local variable bindings in the statements.

The banana brackets are in fact not required. The compiler is intelligent enough to assume that this is what you mean when you write it like this (note that infix notation *is* allowed here):

```
mean4 :: Fractional a => Circuit a a
mean4 = proc value -> do
    (t, n) <- (total -< value) &&& (total -< 1)
    returnA -< t / n
```

So why do we need the banana brackets? For situations where this plainer syntax is ambiguous. The reason is that the arrow part of a `proc` command is *not an ordinary Haskell expression*. Recall that for arrows specified in proc statements, the following things hold true:

- Local variable bindings are only allowed in the input expression after `-<`, and for the `if` and `case` condition. The arrow itself is interpreted in the scope that exists outside `proc`.
- `if` and `case` statements are not plain Haskell. They are implemented using `ArrowChoice`.
- Functions used to combine arrows are not normal Haskell either. They are shorthand for banana bracket notation.

43.7.2 Recursive bindings

At the risk of wearing out the `mean` example, here is yet another way to implement it using recursive bindings. In order for this to work, we'll need an arrow that delays its input by one step:

```
delay :: a -> Circuit a a
delay last = Circuit $ \this -> (delay this, last)
```

Here is what delay does:

```
*Main> runCircuit (delay 0) [5,6,7]
[0,5,6]
*Main>
```

Here is our recursive version of `mean`:

```
mean5 :: Fractional a => Circuit a a
mean5 = proc value -> do
```

```
    rec
        (lastTot, lastN) <- delay (0,0) -< (tot, n)
        let (tot, n) = (lastTot + value, lastN + 1)
        let mean = tot / n
    returnA -< mean
```

The `rec` block resembles a `do`' block, except that

- The last line can be, and usually is, a variable binding. It doesn't matter whether it's a `let` or a `do`-block binding with `<-`.
- The `rec` block doesn't have a return value. `var <- rec ...` is illegal, and `rec` is not allowed to be the last element in a `do` block.
- The use of variables is expected to form a cycle (otherwise there is no point in `rec`).

The machinery of `rec` is handled by the loop function of the `ArrowLoop` class, which we define for Circuit like this:

```
instance ArrowLoop Circuit where
    loop (Circuit cir) = Circuit $ \b ->
        let (cir', (c,d)) = cir (b,d)
        in  (loop cir', c)
```

Behind the scenes, the way it works is this:

- Any variables defined in `rec` that are forward referenced in `rec` are looped around by passing them through the second tuple element of `loop`. Effectively the variable bindings and references to them can be in any order (but the order of arrow statements is significant in terms of effects).
- Any variables defined in `rec` that are referenced from outside `rec` are returned in the first tuple element of `loop`.

It is important to understand that `loop` (and therefore `rec`) simply binds variables. It doesn't hold onto values and pass them back in the next invocation - `delay` does this part. The cycle formed by the variable references must be broken by some sort of delay arrow or lazy evaluation, otherwise the code would die in an infinite loop as if you had written `let a = a+1` in plain Haskell.

43.7.3 ArrowApply

As mentioned before, the arrow part of an arrow statement (before `-<`) can't contain any variables bound inside 'proc'. There is an alternative operator, `-<<` which removes this restriction. It requires the arrow to implement the `ArrowApply` typeclass.

44 Understanding arrows

> **Information**
>
> We have permission to import material from the Haskell arrows page[1]. See the talk page for details.

Arrows, like monads, express computations that happen within a context. However, they are a more general abstraction than monads, and thus allow for contexts beyond what the `Monad` class makes possible. The essential difference between the abstractions can be summed up thus:

> Just as we think of a monadic type m a as representing a ’computation delivering an a ’; so we think of an arrow type a b c, (that is, the application of the parameterised type a to the two parameters b and c) as representing ’a computation with input of type b delivering a c’; arrows make the dependence on input explicit.

This chapter has two main parts. Firstly, we will consider the main ways in which arrow computations differ from those expressed by the functor classes we are used to, and also briefly present some of the core arrow-related type classes. Secondly, we will study the parser example used by John Hughes in the original presentation of arrows.

44.1 Pocket guide to `Arrow`

44.1.1 Arrows look a lot like functions

The first step towards understanding arrows is realising how similar they are to functions. Like `(->)`, the type constructor of an `Arrow` instance has kind `* -> * -> *`, that is, it takes two type arguments – unlike, say, a `Monad`, which takes only one. Crucially, `Arrow` has `Category` as a superclass. `Category` is, to put it very roughly, the class for things that can be composed like functions:

```
class Category y where
    id  :: y a a -- identity for composition.
    (.) :: y b c -> y a b -> y a c -- associative composition.
```

(It goes without saying that functions have an instance of `Category` – in fact, they are `Arrow`s as well.)

A practical consequence of this similarity is that you have to think in point-free terms when looking at expressions full of `Arrow` operators, such as this example from the tutorial:

```
(total &&& (const 1 ^>> total)) >>> arr (uncurry (/))
```

Otherwise you will quickly get lost looking for the values to apply things on. In any case, it is easy to get lost even if you look at such expressions in the right way. That's what `proc` notation is all about: adding extra variable names and whitespace while making some operators implicit, so that `Arrow` code gets easier to follow.

Before continuing, we should mention that Control.Category[2] also defines `(<<<) = (.)` and `(>>>) = flip (.)`, which is very commonly used to compose arrows from left to right.

44.1.2 `Arrow` glides between `Applicative` and `Monad`

In spite of the warning we gave just above, arrows can be compared to applicative functors and monads. The trick is making the functors look more like arrows, and not the opposite. That is, you should not compare `Arrow y => y a b` with `Applicative f => f a` or `Monad m => m a`, but rather with:

- `Applicative f => f (a -> b)`, the type of *static morphisms* i.e. the values to the left of `(<*>)`; and
- `Monad m => a -> m b`, the type of *Kleisli morphisms* i.e. the functions to the right of `(>>=)` [3].

Morphisms are the sort of things that can have `Category` instances, and indeed we could write instances of `Category` for both static and Kleisli morphisms. This modest twisting is enough for a sensible comparison.

If this argument reminds you of the sliding scale of power[4] discussion, in which we compared `Functor`, `Applicative` and `Monad`, that is a sign you are paying attention, as we are following exactly the same route. Back then, we remarked how the types of the morphisms limit how they can, or cannot, create effects. Monadic binds can induce near-arbitrary changes to the effects of a computation depending on the `a` values given to the Kleisli morphism, while the isolation between the functorial wrapper and the function arrow in static morphisms mean the effects in an applicative computation do not depend at all on the values within the functor [5].

What sets arrows apart from this point of view is that in `Arrow y => y a b` there is no such connection between the context `y` and a function arrow to determine so rigidly the range of possibilities. Both static and Kleisli morphisms can be made into `Arrow`s, and conversely an instance of `Arrow` can be made as limited as an `Applicative` one or as powerful as a `Monad` one [6]. More interestingly, we can use `Arrow` to take a third option and have both

[2] `http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Category.html`

[3] Those two concepts are usually known as *static arrows* and *Kleisli arrows* respectively. Since using the word "arrow" with two subtly different meanings would make this text horribly confusing, we opted for "morphism", which is a synonym for this alternative meaning.

[4] `https://en.wikibooks.org/wiki/Haskell%2FApplicative%20functors%20II%23A%20sliding%20scale%20of%20power`

[5] Incidentally, that is why they are called *static*: the effects are set in stone by the sequencing of computations; the generated values cannot affect them.

[6] For details, see *Idioms are oblivious, arrows are meticulous, monads are promiscuous* ^{`http://homepages.inf.ed.ac.uk/wadler/topics/monads.html`} , by Sam Lindley, Philip Wadler and Jeremy Yallop.

applicative-like static effects and monad-like dynamic effects in a single context, but kept separate from each other. Arrows make it possible to fine tune how effects are to be combined. That is the main thrust of the classic example of the arrow-based parser, which we will have a look at near the end of this chapter.

44.1.3 An Arrow can multitask

These are the Arrow methods:

```
class Category y => Arrow y where
    -- Minimal implementation: arr and first
    arr    :: (a -> b) -> y a b                 -- converts function to arrow
    first  :: y a b -> y (a, c) (b, c)          -- maps over first component

    second :: y a b -> y (c, a) (c, b)          -- maps over second component
    (***)  :: y a c -> y b d -> y (a, b) (c, d) -- first and second combined
    (&&&)  :: y a b -> y a c -> y a (b, c)      -- (***) on a duplicated value
```

With these methods, we can carry out multiple computations at each step of what seems to be a linear chain of composed arrows. That is done by keeping values used in separate computations as elements of pairs in a (possibly nested) pair, and then using the using pair-handling functions to reach each value when desired. That allows, for instance, saving intermediate values for later or using functions with multiple arguments conveniently [7].

Visualising may help understanding the data flow in an arrow computation. Here are illustrations of (>>>) and the five Arrow methods:

[7] "Conveniently" is arguably too strong a word, though, given how confusing handling nested tuples can get. *Ergo*, proc notation.

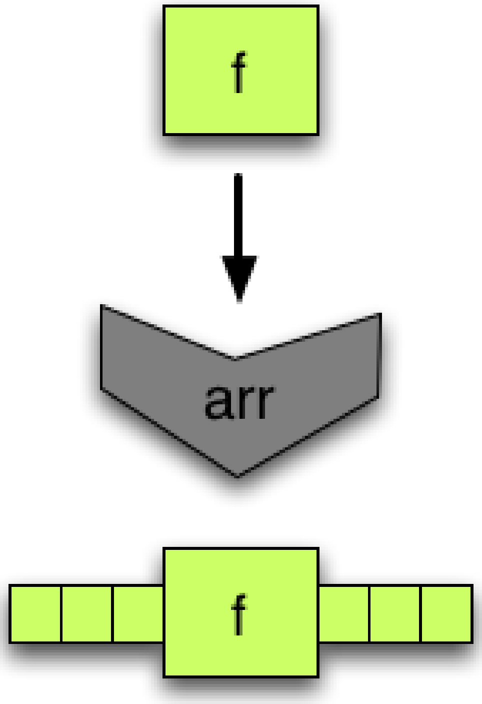


Figure 6 `arr` turns a function into an arrow, composable with other arrows. Naturally, not all arrows are created in this way.

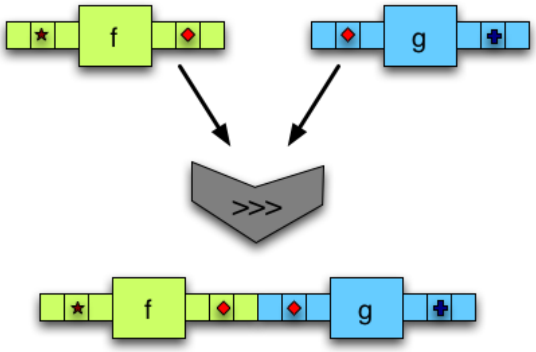


Figure 7 `(>>>)` composes two arrows. The output of the first one is fed to the second.

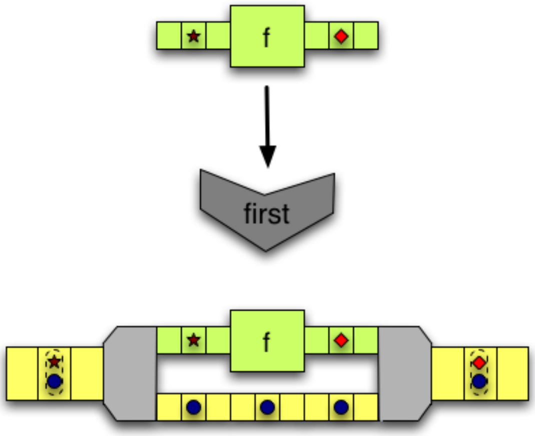


Figure 8 `first` takes two inputs side by side. The first one is modified using an arrow, while the second is left unchanged.

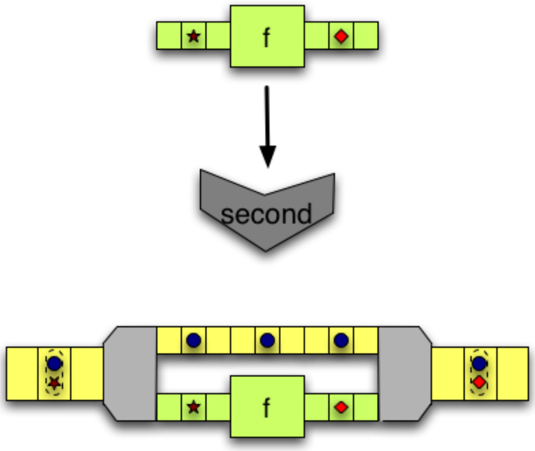


Figure 9 `second`, conversely, takes two inputs but only modifies the second.

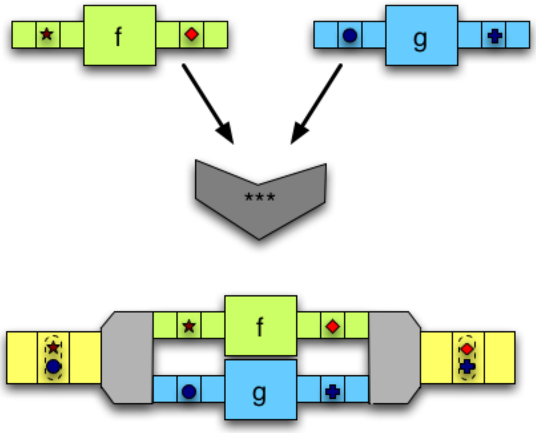


Figure 10 (`***`) takes two inputs and modifies them with two arrows, one for each input.

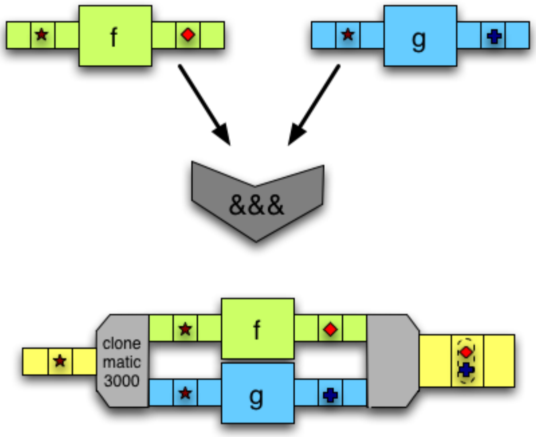


Figure 11 (`&&&`) takes one input, duplicates it and modifies each copy with a different arrow.

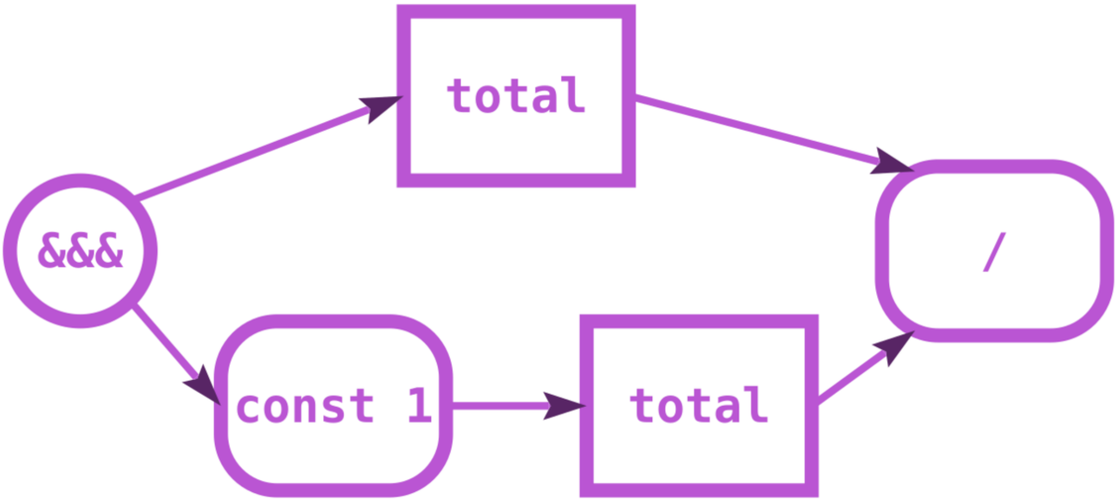


Figure 12 Data flow for the `mean1` arrow from the tutorial[a]. Rectangles are arrows, rounded rectangles are arrows made with `arr`, circles are other data flow split/merge points. Other combinators are left implicit. Corresponding code:

```
(total &&& (const 1 > > total))
    >>> arr (uncurry (/))
```

[a] Chapter 43.3 on page 279

It is worth mentioning that Control.Arrow[8] defines `returnA = arr id` as a do-nothing arrow. One of the arrow laws says `returnA` must be equivalent to the `id` from the `Category` instance [9].

44.1.4 An `ArrowChoice` can be resolute

If `Arrow` makes multitasking possible, `ArrowChoice` forces a decision on what task to do.

```
class Arrow y => ArrowChoice y where
    -- Minimal implementation: left
    left  :: y a b -> y (Either a c) (Either b c)          -- maps over left
 choice

    right :: y a b -> y (Either c a) (Either c b)          -- maps over right
 choice
    (+++) :: y a c -> y b d -> y (Either a b) (Either c d) -- left and right
 combined
    (|||) :: y a c -> y b c -> y (Either a b) c            -- (+++), then merge
 results
```

[8] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Arrow.html

[9] Arrow has laws, and so do the other arrow classes we are discussing in these two chapters. We won't pause to pore over the laws here, but you can check them in the Control.Arrow ^{http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Arrow.html} documentation.

`Either` provides a way to tag the values, so that different arrows can handle them depending on whether they are tagged with `Left` or `Right`. Note that these methods involving `Either` are entirely analogous to those involving pairs offered by `Arrow`.

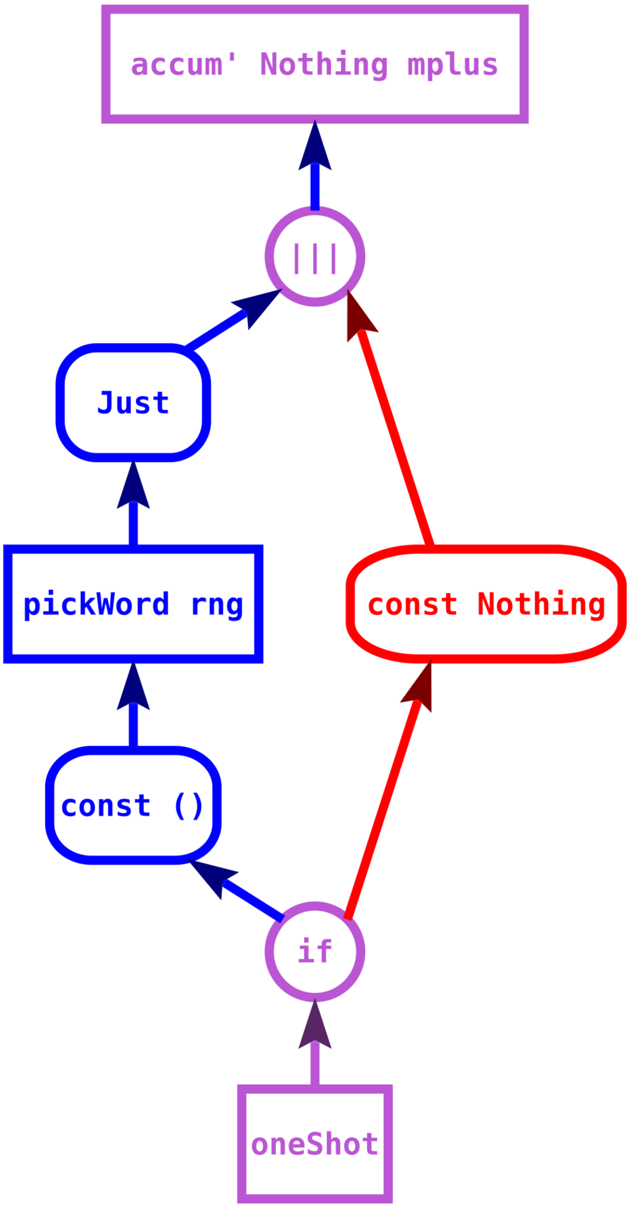


Figure 13 Data flow in a fragment of the `getWord` example of the tutorial[a]. Blue indicates a `Left` tag and red indicates `Right`. Note that the `if` construct of `proc` notation sends `True` to `Left` and `False` to `Right`. Corresponding code:

```
proc () -> do
  firstTime <- oneShot -< ()
  mPicked <- if firstTime
    then do
      picked <- pickWord rng -< ()
      returnA -< Just picked
    else returnA -< Nothing
  accum' Nothing mplus -< mPicked
```

[a] https://en.wikibooks.org/wiki/Haskell%2FArrow%20tutorial%23Hangman%3A%20Pick%20a%20word

44.1.5 An `ArrowApply` is just boring

As the name suggests, `ArrowApply` makes it possible to apply arrows to values directly midway through an arrow computation. Ordinary `Arrow`s do not allow that – we can just compose them on and on and on. Application only happens right at the end, once a run-arrow function of some sort is used to get a plain function from the arrow.

```
class Arrow y => ArrowApply y where
    app :: y (y a b, a) b -- applies first component to second
```

(For instance, `app` for functions is `uncurry ($) = \(f, x) -> f x` .)

`app`, however, comes at a steep price. Building an arrow as a value within an arrow computation and then eliminating it through application implies allowing the values within the computation to affect the context. That sounds a lot like what monadic binds do. It turns out that an `ArrowApply` is *exactly* equivalent to some `Monad` as long as the `ArrowApply` laws are followed. The ultimate consequence is that `ArrowApply` arrows cannot realise any of the interesting possibilities `Arrow` allows but `Monad` doesn't, such as having a partly static context.

> The real flexibility with arrows comes with the ones that aren't monads, otherwise it's just a clunkier syntax.

44.1.6 Arrow combinators crop up in unexpected places

Functions are the trivial example of arrows, and so all of the `Control.Arrow` functions shown above can be used with them. For that reason, it is quite common to see arrow combinators being used in code that otherwise has nothing to do with arrows. Here is a summary of what they do with plain functions, alongside with combinators in other modules that can be used in the same way (in case you prefer the alternative names, or just prefer using simple modules for simple tasks).

Combinator	**What it does (specialised to `(->)`)**	**Alternatives**
`(>>>)`	`flip (.)`	
`first`	`\f (x, y) -> (f x, y)`	`first` (Data.Bifunctor[10])
`second`	`\f (x, y) -> (x, f y)`	`fmap`; `second` (Data.Bifunctor[11])
`(***)`	`\f g (x, y) -> (f x, g y)`	`bimap` (Data.Bifunctor[12])
`(&&&)`	`\f g x -> (f x, g x)`	`liftA2 (,)` (Control.Applicative[13])
`left`	Maps over `Left` case.	`first` (Data.Bifunctor[14])
`right`	Maps over `Right` case.	`fmap`; `second` (Data.Bifunctor[15])

[10] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Bifunctor.html

[11] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Bifunctor.html

[12] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Bifunctor.html

[13] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Applicative.html

[14] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Bifunctor.html

[15] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Bifunctor.html

Combinator	**What it does (specialised to (->))**	**Alternatives**
`(+++)`	Maps over both cases.	`bimap` (Data.Bifunctor[16])
`(|||)`	Eliminates `Either`.	`either` (Data.Either[17])
`app`	`\(f, x) -> f x`	`uncurry ($)`

The `Data.Bifunctor` module provides the `Bifunctor` class, of which pairs and `Either` are instances of. A `Bifunctor` is very much like a `Functor`, except that there are two independent ways of mapping functions over it, corresponding to the `first` and `second` methods [18].

Exercises:

1. Write implementations for `second`, `(***)` and `(&&&)`. Use just `(>>>)`, `arr`, and `first` (plus any plain functions) to implement `second`; after that, you can use the other combinators once you have implemented them.
2. Write an implementation for `right` in terms of `left`.
3. Implement `liftY2 :: Arrow y =>`
   `(a -> b -> c) -> y r a -> y r b -> y r c`

44.2 Using arrows

44.2.1 Avoiding leaks

Arrows were originally motivated by an efficient parser design found by Swierstra & Duponcheel[19].

To describe the benefits of their design, let's examine exactly how monadic parsers work.

If you want to parse a single word, you end up with several monadic parsers stacked end to end. Taking Parsec as an example [20], a parser for the string ”word” can be thought of as [21]:

```
word = do char 'w' >> char 'o' >> char 'r' >> char 'd'
          return "word"
```

[16] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Bifunctor.html

[17] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Either.html

[18] `Data.Bifunctor` was only added to the core GHC libraries in version 7.10, so it might not be installed if you are using an older version. In that case, you can install the `bifunctors` package, which also includes several other bifunctor-related modules

[19] Swierstra, Duponcheel. *Deterministic, error correcting parser combinators.* http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.29.2760

[20] Parsec is a popular and powerful parsing library. See the parsec documentation on Hackage ^{https://hackage.haskell.org/package/parsec} for more information.

[21] ”Thought of as” because in actual code we evidently wouldn't return the string explicitly in such a crude way. Parsec offers a combinator `string` which would allow writing `word = string "word"`. In any case, right now we are only concerned with how characters are tested, and so the crude parser is good enough for a mental model.

Each character is tried in order, if ”worg” is the input, then the first three parsers will succeed, and the last one will fail, making the entire string ”word” parser fail.

If you want to parse one of two options, you create a new parser for each and they are tried in order. The first one must fail in order for the next to be tried with the same input.

```
ab = char 'a' <|> char 'b' <|> char 'c' -- (<|>) is a combinator for
 alternatives.
```

To parse ”c” successfully, both ’a’ and ’b’ must have been tried.

```
one = do char 'o' >> char 'n' >> char 'e'
      return "one"

two = do char 't' >> char 'w' >> char 'o'
      return "two"

three = do char 't' >> char 'h' >> char 'r' >> char 'e' >> char 'e'
        return "three"

nums = one <|> two <|> three
```

With these three parsers, you can’t detect that the string ”four” will fail the parser `nums` until the last parser has failed.

If one of the options can consume much of the input but will fail, you still must descend down the chain of parsers until the final parser fails. All of the input that can possibly be consumed by later parsers must be retained in memory in case one of them does consume it. That can lead to much more space usage than you would naively expect – a situation often called a *space leak*.

The general pattern with monadic parsers, then, is that each option must fail or one option must succeed.

Can it be done better?

Swierstra & Duponcheel (1996) noticed that a smarter parser could immediately fail upon seeing the very first character. For example, in the `nums` parser above, the choice of first letter parsers was limited to either the letter ’o’ for ”one” or the letter ’t’ for both ”two” and ”three”. This smarter parser would also be able to garbage collect input sooner because it could look ahead to see if any other parsers might be able to consume the input, and drop input that could not be consumed. This new parser is a lot like the monadic parsers with the major difference that it exports *static* information. It’s like a monad, but it also tells you what it can parse.

There’s one major problem. This doesn’t fit into the `Monad` interface. Monadic composition works with `(a -> m b)` functions, and functions alone. There’s no way to attach static information. You have only one choice, throw in some input, and see if it passes or fails.

Back when this issue first arose, the monadic interface was being touted as a completely general purpose tool in the functional programming community, so finding that there was some particularly useful code that just couldn’t fit into that interface was something of a setback. This is where arrows come in. John Hughes’s *Generalising monads to arrows* proposed the arrows abstraction as new, more flexible tool.

Static and dynamic parsers

Let us examine Swierstra and Duponcheel's parser in greater detail, from the perspective of arrows a presented by Hughes. The parser has two components: a fast, static parser which tells us if the input is worth trying to parse; and a slow, dynamic parser which does the actual parsing work.

```
import Control.Arrow
import qualified Control.Category as Cat
import Data.List (union)

data Parser s a b = P (StaticParser s) (DynamicParser s a b)
data StaticParser s = SP Bool [s]
newtype DynamicParser s a b = DP ((a, [s]) -> (b, [s]))
```

The static parser consists of a flag, which tells us if the parser can accept the empty input, and a list of possible **starting characters**. For example, the static parser for a single character would be as follows:

```
spCharA :: Char -> StaticParser Char
spCharA c = SP False [c]
```

It does not accept the empty string (`False`) and the list of possible starting characters consists only of `c`.

The dynamic parser needs a little more dissecting. What we see is a function that goes from `(a, [s])` to `(b, [s])`. It is useful to think in terms of sequencing two parsers: each parser consumes the result of the previous parser (`a`), along with the remaining bits of input stream (`[s]`), it does something with `a` to produce its own result `b`, consumes a bit of string and returns *that*. So, as an example of this in action, consider a dynamic parser `(Int, String) -> (Int, String)`, where the `Int` represents a count of the characters parsed so far. The table below shows what would happen if we sequence a few of them together and set them loose on the string ”cake” :

	result	remaining
before	0	cake
after first parser	1	ake
after second parser	2	ke
after third parser	3	e

So the point here is that a dynamic parser has two jobs : it does something to the output of the previous parser (informally, `a -> b`), and it consumes a bit of the input string, (informally, `[s] -> [s]`), hence the type `DP ((a,[s]) -> (b,[s]))`. Now, in the case of a dynamic parser for a single character (type `(Char, String) -> (Char, String)`), the first job is trivial. We ignore the output of the previous parser, return the character we have parsed and consume one character off the stream:

```
dpCharA :: Char -> DynamicParser Char Char Char
dpCharA c = DP (\(_,x:xs) -> (x,xs))
```

This might lead you to ask a few questions. For instance, what's the point of accepting the output of the previous parser if we're just going to ignore it? And shouldn't the dynamic parser be making sure that the current character off the stream matches the character to be parsed by testing `x == c`? The answer to the second question is no – and in fact, this is part of the point: the work is not necessary because the check would already have been performed by the static parser. Naturally, things are only so simple because we are testing just one character. If we were writing a parser for several characters in sequence we would need dynamic parsers that actually tested the second and further characters; and if we wanted to build an output string by chaining several parsers of characters then we would need the output of previous parsers.

Time to put both parsers together. Here is our S+D style parser for a single character:

```
charA :: Char -> Parser Char Char Char
charA c = P (SP False [c]) (DP (\(_,x:xs) -> (x,xs)))
```

Bringing the arrow combinators in

With the preliminary bit of exposition done, we are now going to implement the Arrow class for `Parser s`, and by doing so, give you a glimpse of what makes arrows useful. So let's get started:

```
-- We explain the Eq s constraint below.
instance Eq s => Arrow (Parser s) where
```

`arr` should convert an arbitrary function into a parsing arrow. In this case, we have to use "parse" in a very loose sense: the resulting arrow accepts the empty string, and *only the empty string* (its set of first characters is `[]`). Its sole job is to take the output of the previous parsing arrow and do something with it. That being so, it does not consume any input.

```
    arr f = P (SP True []) (DP (\(b,s) -> (f b,s)))
```

Likewise, the `first` combinator is relatively straightforward. Given a parser, we want to produce a new parser that accepts a pair of inputs `(b,d)`. The first component of the input `b`, is what we actually want to parse. The second part passes through untouched:

```
    first (P sp (DP p)) = P sp (DP (\((b,d),s) ->
        let (c, s') = p (b,s)
        in  ((c,d),s')))
```

We also have to supply the `Category` instance. `id` is entirely obvious, as `id = arr id` must hold:

```
instance Eq s => Cat.Category (Parser s) where
    id = P (SP True []) (DP (\(b,s) -> (b,s)))
    -- Or simply: id = P (SP True []) (DP id)
```

On the other hand, the implementation of `(.)` requires a little more thought. We want to take two parsers, and return a combined parser incorporating the static and dynamic parsers of both arguments:

```
    -- The Eq s constraint is needed for using union here.
    (P (SP empty1 start1) (DP p2)) .
```

```
      (P (SP empty2 start2) (DP p1)) =
        P (SP (empty1 && empty2)
               (if not empty1 then start1 else start1 `union` start2))
          (DP (p2.p1))
```

Combining the dynamic parsers is easy enough; we just do function composition. Putting the static parsers together requires a little bit of thought. First of all, the combined parser can only accept the empty string if *both* parsers do. Fair enough, now how about the starting symbols? Well, the parsers are supposed to be in a sequence, so the starting symbols of the second parser shouldn't really matter. If life were simple, the starting symbols of the combined parser would only be `start1`. Alas, life is not simple, because parsers could very well accept the empty input. If the first parser accepts the empty input, then we have to account for this possibility by accepting the starting symbols from both the first and the second parsers [22].

So what do arrows buy us?

If you look back at our `Parser` type and blank out the static parser section, you might notice that this looks a lot like the arrow instances for functions.

```
arr f = \(b, s) -> (f b, s)
first p = \((b, d), s) ->
            let (c, s') = p (b, s)
            in  ((c, d), s'))

id = id
p2 . p1 = p2 . p1
```

There's the odd `s` variable out for the ride, which makes the definitions look a little strange, but the outline of e.g. the simple `first` functions[23] is there. Actually, what you see here is roughly the arrow instance for the `State` monad/Kleisli morphism (let `f :: b -> c`, `p :: b -> State s c` and `(.)` actually be `(<=<) = flip (>=>)`).

That's fine, but we could have easily done that with bind in classic monadic style, with `first` becoming just an odd helper function that could be easily written with a bit of pattern matching. But remember, our Parser type is not just the dynamic parser – it also contains the static parser.

```
arr f = SP True []
first sp = sp
(SP empty1 start1) >>> (SP empty2 start2) = (SP (empty1 && empty2)
        (if not empty1 then start1 else start1 `union` start2))
```

This is not at all a function, it's just pushing around some data types, and it cannot be expressed in a monadic way. But the `Arrow` interface can deal with just as well. And when we combine the two types, we get a two-for-one deal: the static parser data structure goes along for the ride along with the dynamic parser. The Arrow interface lets us transparently

[22] A reasonable question at this point would be ”Okay, we can compose the static parsers by uniting their lists, but when we are actually gone to test things with them?”. The answer is that the static tests would be performed by the alternatives combinator, which unites two parsers to produce a parser that accepts input from either.

[23] Chapter 44.1.6 on page 296

compose and manipulate the two parsers, static and dynamic, as a unit, which we can then run as a traditional, unified function.

44.3 Arrows in practice

Some examples of libraries using arrows:

- The Haskell XML Toolbox (project page[24] and library documentation[25]) uses arrows for processing XML. There is a Wiki page in the Haskell Wiki with a somewhat Gentle Introduction to HXT[26].
- Netwire (page library documentation[27]) is a library for *functional reactive programming* (FRP). FRP is a functional paradigm for handling events and time-varying values, with use cases including user interfaces, simulations and games. Netwire has an arrow interface as well as an applicative one.
- Yampa (Haskell Wiki page[28] library documentation[29]) is another arrow-based FRP library, and a predecessor to Netwire.
- Hughes' arrow-style parsers were first described in his 2000 paper, but a usable implementation wasn't available until May 2005, when Einar Karttunen released PArrows[30].

44.4 See also

- Bibliography on arrows (haskell.org)[31]

44.5 Acknowledgements

This module uses text from *An Introduction to Arrows* by Shae Erisson, originally written for The Monad.Reader 4

[24] http://www.fh-wedel.de/~si/HXmlToolbox/index.html
[25] https://hackage.haskell.org/package/hxt
[26] http://www.haskell.org/haskellwiki/HXT
[27] http://hackage.haskell.org/package/netwire
[28] https://wiki.haskell.org/Yampa
[29] https://hackage.haskell.org/package/Yampa
[30] https://hackage.haskell.org/package/PArrows
[31] http://www.haskell.org/arrows/biblio.html

45 Continuation passing style (CPS)

Continuation Passing Style (CPS for short) is a style of programming in which functions do not return values; rather, they pass control onto a *continuation*, which specifies what happens next. In this chapter, we are going to consider how that plays out in Haskell and, in particular, how CPS can be expressed with a monad.

45.1 What are continuations?

To dispel puzzlement, we will have a second look at an example from way back in the book, when we introduced the `($)` operator[1]:

```
> map ($ 2) [(2*), (4*), (8*)]
[4,8,16]
```

There is nothing out of ordinary about the expression above, except that it is a little quaint to write that instead of `map (*2) [2, 4, 8]`. The `($)` section makes the code appear backwards, as if we are applying a value to the functions rather than the other way around. And now, the catch: such an innocent-looking reversal is at heart of continuation passing style!

From a CPS perspective, `($ 2)` is a *suspended computation*: a function with general type `(a -> r) -> r` which, given another function as argument, produces a final result. The `a -> r` argument is the *continuation*; it specifies how the computation will be brought to a conclusion. In the example, the functions in the list are supplied as continuations via `map`, producing three distinct results. Note that suspended computations are largely interchangeable with plain values: `flip ($)` [2] converts any value into a suspended computation, and passing `id` as its continuation gives back the original value.

45.1.1 What are they good for?

There is more to continuations than just a parlour trick to impress Haskell newbies. They make it possible to explicitly manipulate, and dramatically alter, the control flow of a program. For instance, returning early from a procedure can be implemented with continuations. Exceptions and failure can also be handled with continuations - pass in a continuation for success, another continuation for fail, and invoke the appropriate continuation. Other possibilities include ”suspending” a computation and returning to it at another

[1] Chapter 19.4 on page 121

[2] That is, `\x -> ($ x)`, fully spelled out as `\x -> \k -> k x`

time, and implementing simple forms of concurrency (notably, one Haskell implementation, Hugs, uses continuations to implement cooperative concurrency).

In Haskell, continuations can be used in a similar fashion, for implementing interesting control flow in monads. Note that there usually are alternative techniques for such use cases, especially in tandem with laziness. In some circumstances, CPS can be used to improve performance by eliminating certain construction-pattern matching sequences (i.e. a function returns a complex structure which the caller will at some point deconstruct), though a sufficiently smart compiler *should* be able to do the elimination [3].

45.2 Passing continuations

An elementary way to take advantage of continuations is to modify our functions so that they return suspended computations rather than ordinary values. We will illustrate how that is done with two simple examples.

45.2.1 `pythagoras`

Example: A simple module, no continuations

```
-- We assume some primitives add and square for the example:

add :: Int -> Int -> Int
add x y = x + y

square :: Int -> Int
square x = x * x

pythagoras :: Int -> Int -> Int
pythagoras x y = add (square x) (square y)
```

Modified to return a suspended computation, `pythagoras` looks like this:

[3] attoparsec ^{http://hackage.haskell.org/package/attoparsec-0.10.4.0/docs/Data-Attoparsec-ByteString.html} is an example of performance-driven usage of CPS.

Example: A simple module, using continuations

```
-- We assume CPS versions of the add and square primitives,
-- (note: the actual definitions of add_cps and square_cps are not
-- in CPS form, they just have the correct type)

add_cps :: Int -> Int -> ((Int -> r) -> r)
add_cps x y = \k -> k (add x y)

square_cps :: Int -> ((Int -> r) -> r)
square_cps x = \k -> k (square x)

pythagoras_cps :: Int -> Int -> ((Int -> r) -> r)
pythagoras_cps x y = \k ->
 square_cps x $ \x_squared ->
 square_cps y $ \y_squared ->
 add_cps x_squared y_squared $ k
```

How the `pythagoras_cps` example works:

1. square x and throw the result into the (\x_squared -> ...) continuation
2. square y and throw the result into the (\y_squared -> ...) continuation
3. add x_squared and y_squared and throw the result into the top level/program continuation `k`.

We can try it out in GHCi by passing `print` as the program continuation:

```
*Main> pythagoras_cps 3 4 print
25
```

If we look at the type of `pythagoras_cps` without the optional parentheses around `(Int -> r) -> r` and compare it with the original type of `pythagoras`, we note that the continuation was in effect added as an extra argument, thus justifying the ”continuation passing style” moniker.

45.2.2 `thrice`

Example: A simple higher order function, no continuations

```
thrice :: (a -> a) -> a -> a
thrice f x = f (f (f x))
```

```
*Main> thrice tail "foobar"
"bar"
```

A higher order function such as `thrice`, when converted to CPS, takes as arguments functions in CPS form as well. Therefore, `f :: a -> a will become`f_cps :: a -> ((a -> r) -> r), and the final type will be `thrice_cps :: (a -> ((a -> r) -> r)) -> a -> ((a -> r) -> r)`. The rest of the definition follows quite naturally from the types - we replace `f` by the CPS version, passing along the continuation at hand.

Example: A simple higher order function, with continuations

```
thrice_cps :: (a -> ((a -> r) -> r)) -> a -> ((a -> r) -> r)
thrice_cps f_cps x = \k ->
 f_cps x $ \fx ->
 f_cps fx $ \ffx ->
 f_cps ffx $ k
```

45.3 The `Cont` monad

Having continuation-passing functions, the next step is providing a neat way of composing them, preferably one which does not require the long chains of nested lambdas we have seen just above. A good start would be a combinator for applying a CPS function to a suspended computation. A possible type for it would be:

```
chainCPS :: ((a -> r) -> r) -> (a -> ((b -> r) -> r)) -> ((b -> r) -> r)
```

(You may want to try implementing it before reading on. Hint: start by stating that the result is a function which takes a `b -> r` continuation; then, let the types guide you.)

And here is the implementation:

```
chainCPS s f = \k -> s $ \x -> f x $ k
```

We supply the original suspended computation `s` with a continuation which makes a new suspended computation (produced by `f`) and passes the final continuation `k` to it. Unsurprisingly, it mirrors closely the nested lambda pattern of the previous examples.

Doesn't the type of `chainCPS` look familiar? If we replace `(a -> r) -> r` with `(Monad m) => m a` and `(b -> r) -> r` with `(Monad m) => m b` we get the `(>>=)` signature. Furthermore, our old friend `flip ($)` plays a `return`-like role, in that it makes a suspended computation out of a value in a trivial way. Lo and behold, we have a monad! All we need now [4] is a `Cont r a` type to wrap suspended computations, with the usual wrapper and unwrapper functions.

```
cont :: ((a -> r) -> r) -> Cont r a
runCont :: Cont r a -> (a -> r) -> r
```

[4] Beyond verifying that the monad laws hold, which is left as an exercise to the reader.

The monad instance for Cont follows directly from our presentation, the only difference being the wrapping and unwrapping cruft:

```
instance Monad (Cont r) where
    return x = cont ($ x)
    s >>= f  = cont $ \c -> runCont s $ \x -> runCont (f x) c
```

The end result is that the monad instance makes the continuation passing (and thus the lambda chains) implicit. The monadic bind applies a CPS function to a suspended computation, and runCont is used to provide the final continuation. For a simple example, the Pythagoras example becomes:

Example: The pythagoras example, using the Cont monad

```
-- Using the Cont monad from the transformers package.
import Control.Monad.Trans.Cont

add_cont :: Int -> Int -> Cont r Int
add_cont x y = return (add x y)

square_cont :: Int -> Cont r Int
square_cont x = return (square x)

pythagoras_cont :: Int -> Int -> Cont r Int
pythagoras_cont x y = do
    x_squared <- square_cont x
    y_squared <- square_cont y
    add_cont x_squared y_squared
```

45.4 callCC

While it is always pleasant to see a monad coming forth naturally, a hint of disappointment might linger at this point. One of the promises of CPS was precise control flow manipulation through continuations. And yet, after converting our functions to CPS we promptly hid the continuations behind a monad. To rectify that, we shall introduce callCC, a function which gives us back explicit control of continuations - but only where we want it.

callCC is a very peculiar function; one that is best introduced with examples. Let us start with a trivial one:

Example: `square` using `callCC`

```
-- Without callCC
square :: Int -> Cont r Int
square n = return (n  2)

-- With callCC
squareCCC  :: Int − > Cont r Int
squareCCC n = callCC $ \k − > k (n  2)
```

The argument passed to `callCC` is a function, whose result is a suspended computation (general type `Cont r a`) which we will refer to as ”the `callCC` computation”. *In principle*, the `callCC` computation is what the whole `callCC` expression evaluates to. The caveat, and what makes `callCC` so special, is due to `k`, the argument to the argument. It is a function which acts as an *eject button*: calling it anywhere will lead to the value passed to it being made into a suspended computation, which then is inserted into control flow at the point of the `callCC` invocation. That happens unconditionally; in particular, whatever follows a `k` invocation in the `callCC` computation is summarily discarded. From another perspective, `k` captures *the rest of the computation* following the `callCC`; calling it throws a value into the continuation at that particular point (”callCC” stands for ”call with current continuation”). While in this simple example the effect is merely that of a plain `return`, `callCC` opens up a number of possibilities, which we are now going to explore.

45.4.1 Deciding when to use `k`

`callCC` gives us extra power over what is thrown into a continuation, and when that is done. The following example begins to show how we can use this extra power.

Example: Our first proper `callCC` function

```
foo :: Int -> Cont r String
foo x = callCC $ \k -> do
    let y = x  2 + 3
    when (y > 20) $ k ”over twenty”
    return (show $ y − 4)
```

`foo` is a slightly pathological function that computes the square of its input and adds three; if the result of this computation is greater than 20, then we return from the `callCC` computation (and, in this case, from the whole function) immediately, throwing the string `"over twenty"` into the continuation that will be passed to `foo`. If not, then we subtract four from our previous computation, `show` it, and throw it into the continuation. Remarkably, `k` here is used just like the ’return’ *statement* from an imperative language, that immediately exits the function. And yet, this being Haskell, `k` is just an ordinary first-class function, so you can pass it to other functions like `when`, store it in a `Reader`, etc.

Naturally, you can embed calls to `callCC` within do-blocks:

Example: More developed `callCC` example involving a do-block

```
bar :: Char -> String -> Cont r Int
bar c s = do
    msg <- callCC $ \k -> do
        let s0 = c : s
        when (s0 == "hello") $ k "They say hello."
        let s1 = show s0
        return ("They appear to be saying " ++ s1)
    return (length msg)
```

When you call `k` with a value, the entire `callCC` call takes that value. In effect, that makes `k` a lot like an 'goto' statement in other languages: when we call `k` in our example, it pops the execution out to where you first called `callCC`, the `msg <- callCC $ ...` line. No more of the argument to `callCC` (the inner do-block) is executed. Hence the following example contains a useless line:

Example: Popping out a function, introducing a useless line

```
quux :: Cont r Int
quux = callCC $ \k -> do
    let n = 5
    k n
    return 25
```

`quux` will return `5`, and not `25`, because we pop out of `quux` before getting to the `return 25` line.

45.4.2 Behind the scenes

We have deliberately broken a trend here: normally when we introduce a function we give its type straight away, but in this case we chose not to. The reason is simple: the type is pretty complex, and it does not immediately give insight into what the function does, or how it works. After the initial presentation of `callCC`, however, we are in a better position to tackle it. Take a deep breath...

```
callCC :: ((a -> Cont r b) -> Cont r a) -> Cont r a
```

We can make sense of that based on what we already know about `callCC`. The overall result type and the result type of the argument have to be the same (i.e. `Cont r a`), as in the absence of an invocation of `k` the corresponding result values are one and the same. Now, what about the type of `k`? As mentioned above, `k`'s argument is made into a suspended computation inserted at the point of the `callCC` invocation; therefore, if the latter has type

`Cont r a` `k`'s argument must have type `a`. As for `k`'s result type, interestingly enough it doesn't matter as long as it is wrapped in the same `Cont r` monad; in other words, the `b` stands for an arbitrary type. That happens because the suspended computation made out of the `a` argument will receive whatever continuation follows the `callCC`, and so the continuation taken by `k`'s result is irrelevant.

Note:
The arbitrariness of `k`'s result type explains why the following variant of the useless line example leads to a type error:

```
quux :: Cont r Int
quux = callCC $ \k -> do
   let n = 5
   when True $ k n
   k 25
```

`k`'s result type *could* be anything of form `Cont r a`; however, the `when` constrains it to `Cont r ()`, and so the closing `k 25` does not match the result type of `quux`. The solution is very simple: replace the final `k` by a plain old `return`.

To conclude this section, here is the implementation of `callCC`. Can you identify `k` in it?

```
callCC f = cont $ \h -> runCont (f (\a -> cont $ \_ -> h a)) h
```

The code is far from obvious. However, the amazing fact is that the implementations of `callCC`, `return` and `(>>=)` for `Cont` can be produced automatically from their type signatures - Lennart Augustsson's Djinn `http://lambda-the-ultimate.org/node/1178` is a program that will do this for you. See Phil Gossett's Google tech talk: `http://www.youtube.com/watch?v=h0OkptwfX4g` for background on the theory behind Djinn; and Dan Piponi's article: `http://www.haskell.org/wikiupload/1/14/TMR-Issue6.pdf` which uses Djinn in deriving continuation passing style.

45.5 Example: a complicated control structure

We will now look at some more realistic examples of control flow manipulation. The first one, presented below, was originally taken from the ”The Continuation monad” section of the All about monads tutorial[5], used with permission.

[5] `http://www.haskell.org/haskellwiki/All_about_monads`

Example: Using Cont for a complicated control structure

```
{- We use the continuation monad to perform "escapes" from code blocks.
This function implements a complicated control structure to process
numbers:

Input (n)     Output                     List Shown
=========     ======                     ==========
0-9           n                          none
10-199        number of digits in (n/2) digits of (n/2)
200-19999     n                          digits of (n/2)
20000-1999999 (n/2) backwards            none
>= 2000000    sum of digits of (n/2)     digits of (n/2)
-}
fun :: Int -> String
fun n = (`runCont` id) $ do
    str <- callCC $ \exit1 -> do                            -- define "exit1"
        when (n < 10) (exit1 (show n))
        let ns = map digitToInt (show (n `div` 2))
        n' <- callCC $ \exit2 -> do                         -- define "exit2"
            when ((length ns) < 3) (exit2 (length ns))
            when ((length ns) < 5) (exit2 n)
            when ((length ns) < 7) $ do
                let ns' = map intToDigit (reverse ns)
                exit1 (dropWhile (=='0') ns')               --escape 2 levels
            return $ sum ns
        return $ "(ns = " ++ (show ns) ++ ") " ++ (show n')
    return $ "Answer: " ++ str
```

`fun` is a function that takes an integer `n`. The implementation uses `Cont` and `callCC` to set up a control structure using `Cont` and `callCC` that does different things based on the range that `n` falls in, as stated by the comment at the top. Let us dissect it:

1. Firstly, the ``(`runCont` id)`` at the top just means that we run the `Cont` block that follows with a final continuation of `id` (or, in other words, we extract the value from the suspended computation unchanged). That is necessary as the result type of `fun` doesn't mention `Cont`.
2. We bind `str` to the result of the following `callCC` do-block:
   a) If `n` is less than 10, we exit straight away, just showing `n`.
   b) If not, we proceed. We construct a list, `ns`, of digits of ``n `div` 2``.
   c) `n'` (an `Int`) gets bound to the result of the following inner `callCC` do-block.
      i. If `length ns < 3`, i.e., if ``n `div` 2`` has less than 3 digits, we pop out of this inner do-block with the number of digits as the result.
      ii. If ``n `div` 2`` has less than 5 digits, we pop out of the inner do-block returning the original `n`.

        iii. If `n `div` 2` has less than 7 digits, we pop out of *both* the inner and outer do-blocks, with the result of the digits of `n `div` 2` in reverse order (a `String`).
        iv. Otherwise, we end the inner do-block, returning the sum of the digits of `n `div` 2`.
    d) We end this do-block, returning the String `"(ns = X) Y"`, where X is `ns`, the digits of `n `div` 2`, and Y is the result from the inner do-block, `n'`.
3. Finally, we return out of the entire function, with our result being the string ”Answer: Z”, where Z is the string we got from the `callCC` do-block.

45.6 Example: exceptions

One use of continuations is to model exceptions. To do this, we hold on to *two* continuations: one that takes us out to the handler in case of an exception, and one that takes us to the post-handler code in case of a success. Here's a simple function that takes two numbers and does integer division on them, failing when the denominator is zero.

Example: An exception-throwing `div`

```
divExcpt :: Int -> Int -> (String -> Cont r Int) -> Cont r Int
divExcpt x y handler = callCC $ \ok -> do
    err <- callCC $ \notOk -> do
        when (y == 0) $ notOk "Denominator 0"
        ok $ x `div` y
    handler err

{- For example,
runCont (divExcpt 10 2 error) id --> 5
runCont (divExcpt 10 0 error) id --> *** Exception: Denominator 0
-}
```

How does it work? We use two nested calls to `callCC`. The first labels a continuation that will be used when there's no problem. The second labels a continuation that will be used when we wish to throw an exception. If the denominator isn't 0, `x `div` y` is thrown into the `ok` continuation, so the execution pops right back out to the top level of `divExcpt`. If, however, we were passed a zero denominator, we throw an error message into the `notOk` continuation, which pops us out to the inner do-block, and that string gets assigned to `err` and given to `handler`.

A more general approach to handling exceptions can be seen with the following function. Pass a computation as the first parameter (more precisely, a function which takes an error-throwing function and results in the computation) and an error handler as the second parameter. This example takes advantage of the generic `MonadCont` class [6] which covers both `Cont` and the corresponding `ContT` transformer by default, as well as any other continuation monad which instantiates it.

Example: General `try` using continuations.

```
import Control.Monad.Cont

tryCont :: MonadCont m => ((err -> m a) -> m a) -> (err -> m a) -> m a
tryCont c h = callCC $ \ok -> do
    err <- callCC $ \notOk -> do
        x <- c notOk
        ok x
    h err
```

And here is our `try` in action:

Example: Using `try`

```
data SqrtException = LessThanZero deriving (Show, Eq)

sqrtIO :: (SqrtException -> ContT r IO ()) -> ContT r IO ()
sqrtIO throw = do
    ln <- lift (putStr "Enter a number to sqrt: " >> readLn)
    when (ln < 0) (throw LessThanZero)
    lift $ print (sqrt ln)

main = runContT (tryCont sqrtIO (lift . print)) return
```

In this example, error throwing means escaping from an enclosing `callCC`. The `throw` in `sqrtIO` jumps out of `tryCont`'s inner `callCC`.

45.7 Example: coroutines

In this section we make a CoroutineT monad that provides a monad with `fork`, which enqueues a new suspended coroutine, and `yield`, that suspends the current thread.

```
{-# LANGUAGE GeneralizedNewtypeDeriving #-}
-- We use GeneralizedNewtypeDeriving to avoid boilerplate. As of GHC 7.8, it is
 safe.
```

[6] Found in the `mtl` package, module Control.Monad.Cont ^{`http://hackage.haskell.org/packages/archive/mtl/2.1.2/doc/html/Control-Monad-Cont.html`} .

```
import Control.Applicative
import Control.Monad.Cont
import Control.Monad.State

-- The CoroutineT monad is just ContT stacked with a StateT containing the
 suspended coroutines.
newtype CoroutineT r m a = CoroutineT {runCoroutineT' :: ContT r (StateT
 [CoroutineT r m ()] m) a}
    deriving (Functor,Applicative,Monad,MonadCont,MonadIO)

-- Used to manipulate the coroutine queue.
getCCs :: Monad m => CoroutineT r m [CoroutineT r m ()]
getCCs = CoroutineT $ lift get

putCCs :: Monad m => [CoroutineT r m ()] -> CoroutineT r m ()
putCCs = CoroutineT . lift . put

-- Pop and push coroutines to the queue.
dequeue :: Monad m => CoroutineT r m ()
dequeue = do
    current_ccs <- getCCs
    case current_ccs of
        [] -> return ()
        (p:ps) -> do
            putCCs ps
            p

queue :: Monad m => CoroutineT r m () -> CoroutineT r m ()
queue p = do
    ccs <- getCCs
    putCCs (ccs++[p])

-- The interface.
yield :: Monad m => CoroutineT r m ()
yield = callCC $ \k -> do
    queue (k ())
    dequeue

fork :: Monad m => CoroutineT r m () -> CoroutineT r m ()
fork p = callCC $ \k -> do
    queue (k ())
    p
    dequeue

-- Exhaust passes control to suspended coroutines repeatedly until there isn't
 any left.
exhaust :: Monad m => CoroutineT r m ()
exhaust = do
    exhausted <- null <$> getCCs
    if not exhausted
        then yield >> exhaust
        else return ()

-- Runs the coroutines in the base monad.
runCoroutineT :: Monad m => CoroutineT r m r -> m r
runCoroutineT = flip evalStateT [] . flip runContT return . runCoroutineT' . (<*
 exhaust)
```

Some example usage:

```
printOne n = do
    liftIO (print n)
    yield

example = runCoroutineT $ do
    fork $ replicateM_ 3 (printOne 3)
    fork $ replicateM_ 4 (printOne 4)
    replicateM_ 2 (printOne 2)
```

Outputting:

```
3
4
3
2
4
3
2
4
4
```

45.8 Example: Implementing pattern matching

An interesting usage of CPS functions is to implement our own pattern matching. We will illustrate how this can be done by some examples.

Example: Built-in pattern matching

```
check :: Bool -> String
check b = case b of
    True  -> "It's True"
    False -> "It's False"
```

Now we have learnt CPS, we can refactor the code like this.

Example: Pattern matching in CPS

```
type BoolCPS r = r -> r -> r

true :: BoolCPS r
true x _ = x

false :: BoolCPS r
false _ x = x

check :: BoolCPS String -> String
check b = b "It's True" "It's False"
```

```
*Main> check true
"It's True"
*Main> check false
"It's False"
```

What happens here is that, instead of plain values, we represent `True` and `False` by functions that would choose either the first or second argument they are passed. Since `true` and `false` behave differently, we can achieve the same effect as pattern matching. Furthermore, `True`, `False` and `true`, `false` can be converted back and forth by `\b -> b True False` and `\b -> if b then true else false`.

We should see how this is related to CPS in this more complicated example.

Example: More complicated pattern matching and its CPS equivalence

```
data Foobar = Zero | One Int | Two Int Int

type FoobarCPS r = r -> (Int -> r) -> (Int -> Int -> r) -> r

zero :: FoobarCPS r
zero x _ _ = x

one :: Int -> FoobarCPS r
one x _ f _ = f x

two :: Int -> Int -> FoobarCPS r
two x y _ _ f = f x y

fun :: Foobar -> Int
fun x = case x of
    Zero -> 0
    One a -> a + 1
    Two a b -> a + b + 2

funCPS :: FoobarCPS Int -> Int
funCPS x = x 0 (+1) (\a b -> a + b + 2)
```

```
*Main> fun zero
0
*Main> fun $ one 3
4
*Main> fun $ two 3 4
9
```

Similar to former example, we represent values by functions. These function-values pick the corresponding (i.e. match) continuations they are passed to and pass to the latter the values stored in the former. An interesting thing is that this process involves in no comparison. As we know, pattern matching can work on types that are not instances of `Eq`: the function-values "know" what their patterns are and would automatically pick the right continuations. If this is done from outside, say, by an `pattern_match :: [(pattern, result)] -> value -> result` function, it would have to inspect and compare the patterns and the values to see if they match -- and thus would need `Eq` instances.

46 Zippers

46.1 Theseus and the Zipper

46.1.1 The Labyrinth

”Theseus, we have to do something” said Homer, chief marketing officer of Ancient Geeks Inc.. Theseus put the Minotaur action figure™ back onto the shelf and nodded. ”Today’s children are no longer interested in the ancient myths, they prefer modern heroes like Spiderman or Sponge Bob.” *Heroes.* Theseus knew well how much he had been a hero in the labyrinth back then on Crete[1]. But those ”modern heroes” did not even try to appear realistic. What made them so successful? Anyway, if the pending sales problems could not be resolved, the shareholders would certainly arrange a passage over the Styx for Ancient Geeks Inc.

”Heureka! Theseus, I have an idea: we implement your story with the Minotaur as a computer game! What do you say?” Homer was right. There had been several books, epic (and chart breaking) songs, a mandatory movie trilogy and uncountable Theseus & the Minotaur™ gimmicks, but a computer game was missing. ”Perfect, then. Now, Theseus, your task is to implement the game”.

A true hero, Theseus chose Haskell as the language to implement the company’s redeeming product in. Of course, exploring the labyrinth of the Minotaur was to become one of the game’s highlights. He pondered: ”We have a two-dimensional labyrinth whose corridors can point in many directions. Of course, we can abstract from the detailed lengths and angles: for the purpose of finding the way out, we only need to know how the path forks. To keep things easy, we model the labyrinth as a tree. This way, the two branches of a fork cannot join again when walking deeper and the player cannot go round in circles. But I think there is enough opportunity to get lost; and this way, if the player is patient enough, he can explore the entire labyrinth with the left-hand rule.”

```
data Node a = DeadEnd a
            | Passage a (Node a)
            | Fork    a (Node a) (Node a)
```

[1] Ian Stewart. *The true story of how Theseus found his way out of the labyrinth.* Scientific American, February 1991, page 137.

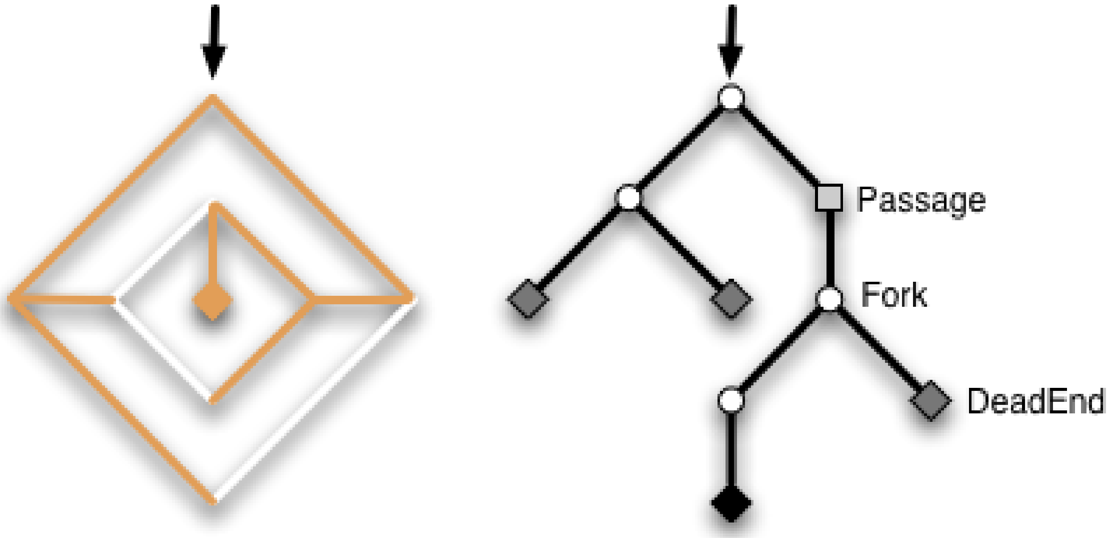


Figure 14 An example labyrinth and its representation as tree.

Theseus made the nodes of the labyrinth carry an extra parameter of type a. Later on, it may hold game relevant information like the coordinates of the spot a node designates, the ambience around it, a list of game items that lie on the floor, or a list of monsters wandering in that section of the labyrinth. We assume that two helper functions

```
get :: Node a -> a
put :: a -> Node a -> Node a
```

retrieve and change the value of type `a` stored in the first argument of every constructor of `Node a`.

Exercises:

1. Implement `get` and `put`. One case for `get` is
   `get (Passage x _) = x`.
2. To get a concrete example, write down the labyrinth shown in the picture as a value of type `Node (Int,Int)`. The extra parameter `(Int,Int)` holds the cartesian coordinates of a node.

"Mh, how to represent the player's current position in the labyrinth? The player can explore deeper by choosing left or right branches, like in"

```
turnRight :: Node a -> Maybe (Node a)
turnRight (Fork _ l r) = Just r
turnRight _            = Nothing
```

"But replacing the current top of the labyrinth with the corresponding sub-labyrinth this way is not an option, because he cannot go back then." He pondered. "Ah, we can apply *Ariadne's trick with the thread* for going back. We simply represent the player's position by the list of branches his thread takes, the labyrinth always remains the same.”

```
data Branch = KeepStraightOn
            | TurnLeft
            | TurnRight
type Thread = [Branch]
```

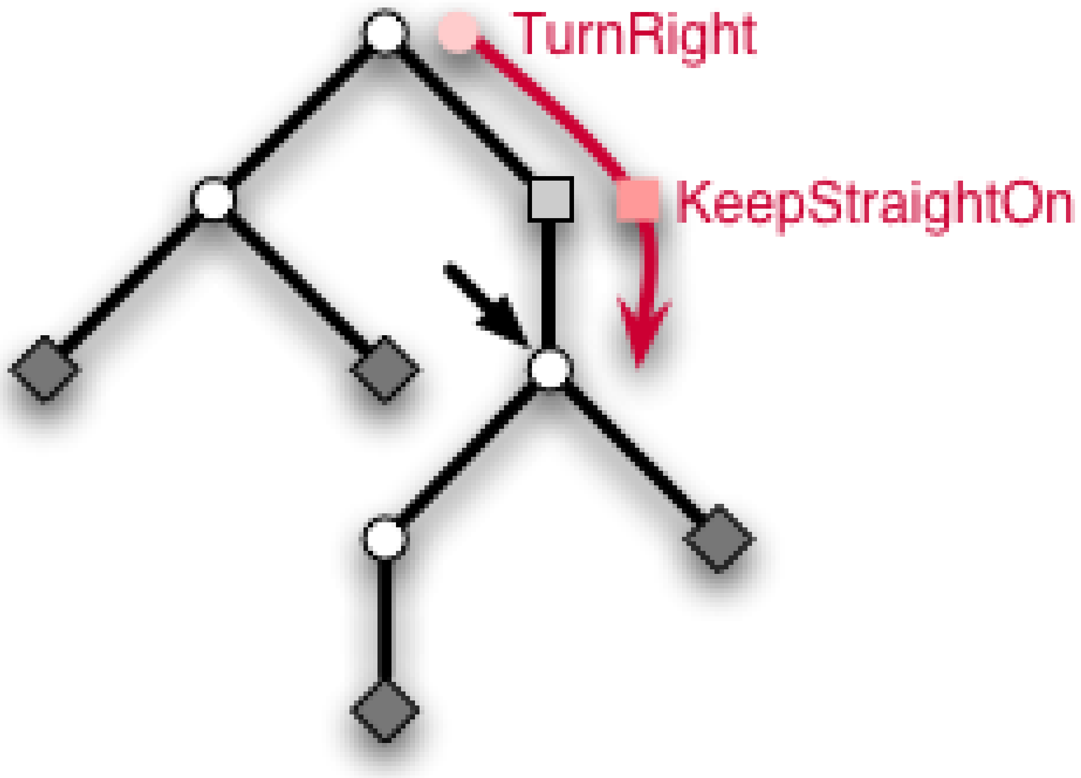


Figure 15 Representation of the player’s position by Ariadne’s thread.

”For example, a thread `[TurnRight,KeepStraightOn]` means that the player took the right branch at the entrance and then went straight down a `Passage` to reach its current position. With the thread, the player can now explore the labyrinth by extending or shortening it. For instance, the function `turnRight` extends the thread by appending the `TurnRight` to it.”

```
turnRight :: Thread -> Thread
turnRight t = t ++ [TurnRight]
```

”To access the extra data, i.e. the game relevant items and such, we simply follow the thread into the labyrinth.”

```
retrieve :: Thread -> Node a -> a
retrieve []                  n             = get n
retrieve (KeepStraightOn:bs) (Passage _ n) = retrieve bs n
retrieve (TurnLeft      :bs) (Fork _ l r)  = retrieve bs l
retrieve (TurnRight     :bs) (Fork _ l r)  = retrieve bs r
```

Exercises:
Write a function `update` that applies a function of type `a -> a` to the extra data at the player's position.

Theseus' satisfaction over this solution did not last long. ”Unfortunately, if we want to extend the path or go back a step, we have to change the last element of the list. We could store the list in reverse, but even then, we have to follow the thread again and again to access the data in the labyrinth at the player's position. Both actions take time proportional to the length of the thread and for large labyrinths, this will be too long. Isn't there another way?”

46.1.2 Ariadne's Zipper

While Theseus was a skillful warrior, he did not train much in the art of programming and could not find a satisfying solution. After intense but fruitless cogitation, he decided to call his former love Ariadne to ask her for advice. After all, it was she who had the idea with the thread. Ariadne Consulting. What can I do for you? Our hero immediately recognized the voice. ”Hello Ariadne, it's Theseus.” An uneasy silence paused the conversation. Theseus remembered well that he had abandoned her on the island of Naxos and knew that she would not appreciate his call. But Ancient Geeks Inc. was on the road to Hades and he had no choice. ”Uhm, darling, ... how are you?” Ariadne retorted an icy response, Mr. Theseus, the times of *darling* are long over. What do you want? ”Well, I uhm ... I need some help with a programming problem. I'm programming a new Theseus & the Minotaur™ computer game.” She jeered, Yet another artifact to glorify your 'heroic being'? And you want me of all people to help you? ”Ariadne, please, I beg of you, Ancient Geeks Inc. is on the brink of insolvency. The game is our last hope!” After a pause, she came to a decision. Fine, I will help you. But only if you transfer a substantial part of Ancient Geeks Inc. to me. Let's say thirty percent. Theseus turned pale. But what could he do? The situation was desperate enough, so he agreed but only after negotiating Ariadne's share to a tenth.

After Theseus told Ariadne of the labyrinth representation he had in mind, she could immediately give advice, You need a **zipper**. ”Huh? What does the problem have to do with my fly?” Nothing, it's a data structure first published by Gérard Huet[2]. ”Ah.” More precisely, it's a purely functional way to augment tree-like data structures like lists or binary trees with a single **focus** or **finger** that points to a subtree inside the data structure and allows constant time updates and lookups at the spot it points to[3]. In our

[2] Gérard Huet. *The Zipper.* Journal of Functional Programming, 7 (5), Sept 1997, pp. 549--554. PDF ^{http://www.st.cs.uni-sb.de/edu/seminare/2005/advanced-fp/docs/huet-zipper.pdf}

[3] Note the notion of *zipper* as coined by Gérard Huet also allows to replace whole subtrees even if there is no extra data associated with them. In the case of our labyrinth, this is irrelevant. We will come back to this in the section Differentiation of data types ^{Chapter46.2 on page 326}.

case, we want a focus on the player's position. "I know for myself that I want fast updates, but how do I code it?" Don't get impatient, you cannot solve problems by coding, you can only solve them by thinking. The only place where we can get constant time updates in a purely functional data structure is the topmost node[4][5]. So, the focus necessarily has to be at the top. Currently, the topmost node in your labyrinth is always the entrance, but your previous idea of replacing the labyrinth by one of its sub-labyrinths ensures that the player's position is at the topmost node. "But then, the problem is how to go back, because all those sub-labyrinths get lost that the player did not choose to branch into." Well, you can use my thread in order not to lose the sub-labyrinths. Ariadne savored Theseus' puzzlement but quickly continued before he could complain that he already used Ariadne's thread, The key is to *glue the lost sub-labyrinths to the thread* so that they actually don't get lost at all. The intention is that the thread and the current sub-labyrinth complement one another to the whole labyrinth. With 'current' sub-labyrinth, I mean the one that the player stands on top of. The zipper simply consists of the thread and the current sub-labyrinth.

```
type Zipper a = (Thread a, Node a)
```

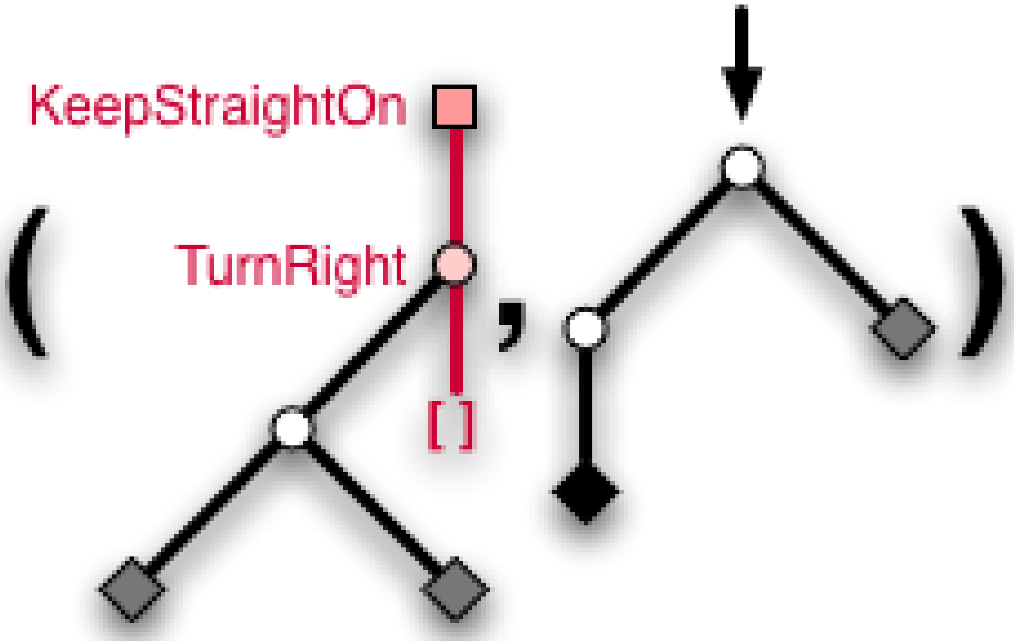


Figure 16 The zipper is a pair of Ariadne's thread and the current sub-labyrinth that the player stands on top. The main thread is colored red and has sub-labyrinths attached to it, such that the whole labyrinth can be reconstructed from the pair.

[4] Of course, the second topmost node or any other node at most a constant number of links away from the top will do as well.

[5] Note that changing the whole data structure as opposed to updating the data at a node can be achieved in amortized constant time even if more nodes than just the top node is affected. An example is incrementing a number in binary representation. While incrementing say `111..11` must touch all digits to yield `1000..00`, the increment function nevertheless runs in constant amortized time (but not in constant worst case time).

Theseus didn't say anything. You can also view the thread as a **context** in which the current sub-labyrinth resides. Now, let's find out how to define `Thread a`. By the way, `Thread` has to take the extra parameter `a` because it now stores sub-labyrinths. The thread is still a simple list of branches, but the branches are different from before.

```
data Branch a  = KeepStraightOn a
               | TurnLeft  a (Node a)
               | TurnRight a (Node a)
type Thread a  = [Branch a]
```

Most importantly, `TurnLeft` and `TurnRight` have a sub-labyrinth glued to them. When the player chooses say to turn right, we extend the thread with a `TurnRight` and now attach the untaken left branch to it, so that it doesn't get lost. Theseus interrupts, ”Wait, how would I implement this behavior as a function `turnRight`? And what about the first argument of type `a` for `TurnRight`? Ah, I see. We not only need to glue the branch that would get lost, but also the extra data of the `Fork` because it would otherwise get lost as well. So, we can generate a new branch by a preliminary”

```
branchRight (Fork x l r) = TurnRight x l
```

”Now, we have to somehow extend the existing thread with it.” Indeed. The second point about the thread is that it is stored *backwards.* To extend it, you put a new branch in front of the list. To go back, you delete the topmost element. ”Aha, this makes extending and going back take only constant time, not time proportional to the length as in my previous version. So the final version of `turnRight` is”

```
turnRight :: Zipper a -> Maybe (Zipper a)
turnRight (t, Fork x l r) = Just (TurnRight x l : t, r)
turnRight _               = Nothing
```

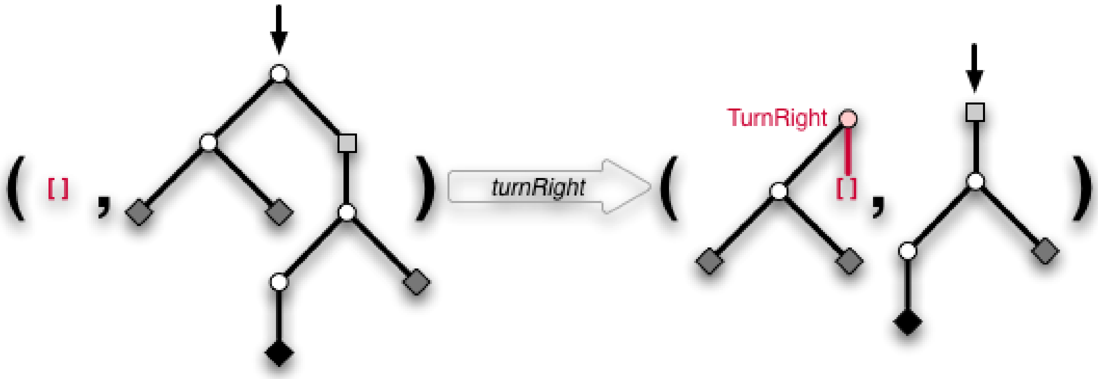


Figure 17 Taking the right subtree from the entrance. Of course, the thread is initially empty. Note that the thread runs backwards, i.e. the topmost segment is the most recent.

”That was not too difficult. So let's continue with `keepStraightOn` for going down a passage. This is even easier than choosing a branch as we only need to keep the extra data:"

```
keepStraightOn :: Zipper a -> Maybe (Zipper a)
keepStraightOn (t, Passage x n) = Just (KeepStraightOn x : t, n)
keepStraightOn _                = Nothing
```

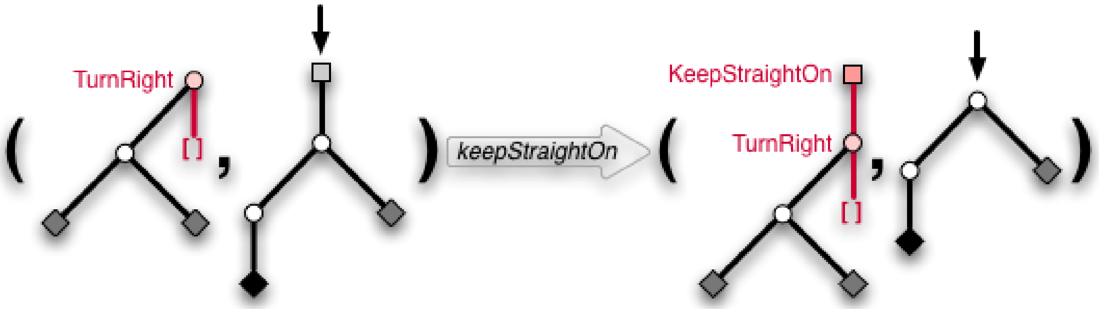


Figure 18 Now going down a passage.

Exercises:
Write the function `turnLeft`.

Pleased, he continued, "But the interesting part is to go back, of course. Let's see..."

```
back :: Zipper a -> Maybe (Zipper a)
back ([]                    , _) = Nothing
back (KeepStraightOn x : t , n) = Just (t, Passage x n)
back (TurnLeft  x r    : t , l) = Just (t, Fork x l r)
back (TurnRight x l    : t , r) = Just (t, Fork x l r)
```

"If the thread is empty, we're already at the entrance of the labyrinth and cannot go back. In all other cases, we have to wind up the thread. And thanks to the attachments to the thread, we can actually reconstruct the sub-labyrinth we came from." Ariadne remarked, Note that a partial test for correctness is to check that each bound variable like `x`, `l` and `r` on the left hand side appears exactly once at the right hands side as well. So, when walking up and down a zipper, we only redistribute data between the thread and the current sub-labyrinth.

Exercises:

1. Now that we can navigate the zipper, code the functions `get`, `put` and `update` that operate on the extra data at the player's position.
2. Zippers are by no means limited to the concrete example `Node a`, they can be constructed for all tree-like data types. Go on and construct a zipper for binary trees

   ```
   data Tree a = Leaf a | Bin (Tree a) (Tree a)
   ```

   Start by thinking about the possible branches `Branch a` that a thread can take. What do you have to glue to the thread when exploring the tree?
3. Simple lists have a zipper as well.

   ```
   data List a = Empty | Cons a (List a)
   ```

   What does it look like?
4. Write a complete game based on Theseus' labyrinth.

Heureka! That was the solution Theseus sought and Ancient Geeks Inc. should prevail, even if partially sold to Ariadne Consulting. But one question remained: "Why is it called zipper?" Well, I would have called it 'Ariadne's pearl necklace'. But most likely, it's called zipper because the thread is in analogy to the open part and the sub-labyrinth is like the closed part of a zipper. Moving around in the data structure is analogous to zipping or unzipping the zipper. "'Ariadne's pearl necklace'," he articulated disdainfully. "As if your thread was any help back then on Crete." As if the idea with the thread were yours, she replied. "Bah, I need no thread," he defied the fact that he actually did need the thread to program the game. Much to his surprise, she agreed, Well, indeed you don't need a thread. Another view is to literally grab the tree at the focus with your finger and lift it up in the air. The focus will be at the top and all other branches of the tree hang down. You only have to assign the resulting tree a suitable algebraic data type, most likely that of the zipper.

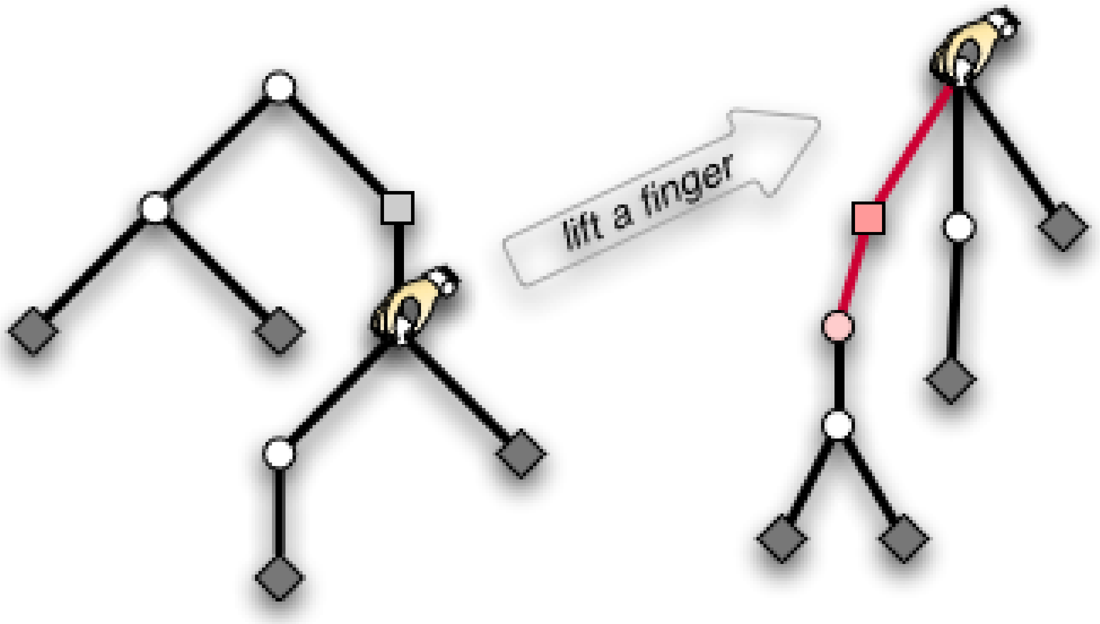


Figure 19 Grab the focus with your finger, lift it in the air and the hanging branches will form new tree with your finger at the top, ready to be structured by an algebraic data type.

”Ah.” He didn’t need Ariadne’s thread but he needed Ariadne to tell him? That was too much. ”Thank you, Ariadne, good bye.” She did not hide her smirk as he could not see it anyway through the phone.

Exercises:
Take a list, fix one element in the middle with your finger and lift the list into the air. What type can you give to the resulting tree?

Half a year later, Theseus stopped in front of a shop window, defying the cold rain that tried to creep under his buttoned up anorak. Blinking letters announced

”Spider-Man: lost in the Web”

- find your way through the labyrinth of threads -

the great computer game by Ancient Geeks Inc.

He cursed the day when he called Ariadne and sold her a part of the company. Was it she who contrived the unfriendly takeover by WineOS Corp., led by Ariadne’s husband Dionysus? Theseus watched the raindrops finding their way down the glass window. After the production line was changed, nobody would produce Theseus and the Minotaur™ merchandise anymore. He sighed. His time, the time of heroes, was over. Now came the super-heroes.

46.2 Differentiation of data types

The previous section has presented the zipper, a way to augment a tree-like data structure `Node a` with a finger that can focus on the different subtrees. While we constructed a zipper for a particular data structure `Node a`, the construction can be easily adapted to different tree data structures by hand.

Exercises:
Start with a ternary tree

```
data Tree a = Leaf a | Node (Tree a) (Tree a) (Tree a)
```

and derive the corresponding `Thread a` and `Zipper a`.

46.2.1 Mechanical Differentiation

But there is also an entirely mechanical way to derive the zipper of any (suitably regular) data type. Surprisingly, 'derive' is to be taken literally, for the zipper can be obtained by the **derivative** of the data type, a discovery first described by Conor McBride[6]. The subsequent section is going to explicate this truly wonderful mathematical gem.

For a systematic construction, we need to calculate with types. The basics of structural calculations with types are outlined in a separate chapter ../Generic Programming/[7] and we will heavily rely on this material.

Let's look at some examples to see what their zippers have in common and how they hint differentiation. The type of binary tree is the fixed point of the recursive equation

$$\mathit{Tree2} = 1 + \mathit{Tree2} \times \mathit{Tree2}$$

.

When walking down the tree, we iteratively choose to enter the left or the right subtree and then glue the not-entered subtree to Ariadne's thread. Thus, the branches of our thread have the type

$$\mathit{Branch2} = \mathit{Tree2} + \mathit{Tree2} \cong 2 \times \mathit{Tree2}$$

.

Similarly, the thread for a ternary tree

$$\mathit{Tree3} = 1 + \mathit{Tree3} \times \mathit{Tree3} \times \mathit{Tree3}$$

[6] Conor Mc Bride. *The Derivative of a Regular Type is its Type of One-Hole Contexts.* Available online. PDF ^{http://strictlypositive.org/diff.pdf}

[7] https://en.wikibooks.org/wiki/..%2FGeneric%20Programming%2F

has branches of type

$$\mathit{Branch3} = 3 \times \mathit{Tree3} \times \mathit{Tree3}$$

because at every step, we can choose between three subtrees and have to store the two subtrees we don't enter. Isn't this strikingly similar to the derivatives $\frac{d}{dx}x^2 = 2 \times x$ and $\frac{d}{dx}x^3 = 3 \times x^2$?

The key to the mystery is the notion of the **one-hole context** of a data structure. Imagine a data structure parameterised over a type X, like the type of trees $\mathit{Tree}\,X$. If we were to remove one of the items of this type X from the structure and somehow mark the now empty position, we obtain a structure with a marked hole. The result is called "one-hole context" and inserting an item of type X into the hole gives back a completely filled $\mathit{Tree}\,X$. The hole acts as a distinguished position, a focus. The figures illustrate this.

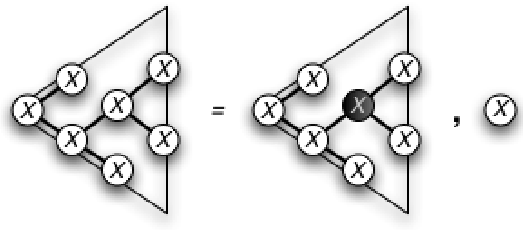


Figure 20 Removing a value of type X from a $\mathit{Tree}\,X$ leaves a hole at that position.

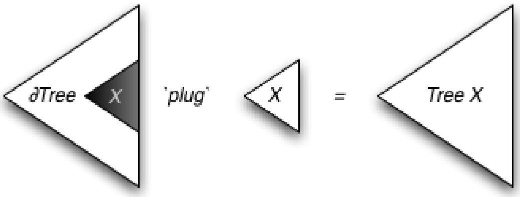
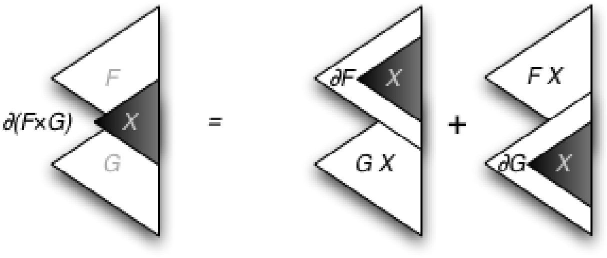
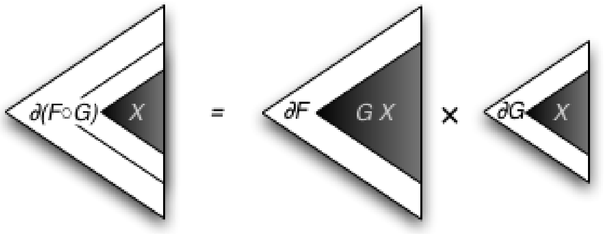


Figure 21 A more abstract illustration of plugging X into a one-hole context.

Of course, we are interested in the type to give to a one-hole context, i.e. how to represent it in Haskell. The problem is how to efficiently mark the focus. But as we will see, finding a representation for one-hole contexts by induction on the structure of the type we want to take the one-hole context of automatically leads to an efficient data type[8]. So, given a data structure $F\,X$ with a functor F and an argument type X, we want to calculate the type $\partial F\,X$ of one-hole contexts from the structure of F. As our choice of notation ∂F already reveals, the rules for constructing one-hole contexts of sums, products and compositions are exactly Leibniz' rules for differentiation.

One-hole context		**Illustration**
$(\partial \mathit{Const}_A)\,X$	$= 0$	There is no X in $A = \mathit{Const}_A\,X$, so the type of its one-hole contexts must be empty.
$(\partial \mathit{Id})\,X$	$= 1$	There is only one position for items X in $X = \mathit{Id}\,X$. Removing one X leaves no X in the result. And as there is only one position we can remove it from, there is exactly one one-hole context for $\mathit{Id}\,X$. Thus, the type of one-hole contexts is the singleton type.
$\partial(F + G)$	$= \partial F + \partial G$	As an element of type $F + G$ is either of type F or of type G, a one-hole context is also either ∂F or ∂G.

[8] This phenomenon already shows up with generic tries.

One-hole context		Illustration
$\partial(F\times G)$	$= F\times\partial G+\partial F\times G$	 F ∂(F×G) X G = ∂F X G X + F X ∂G X **Figure 22** The hole in a one-hole context of a pair is either in the first or in the second component.
$\partial(F\circ G)$	$= (\partial F\circ G)\times\partial G$	 **Figure 23** **Chain rule**. The hole in a composition arises by making a hole in the enclosing structure and fitting the enclosed structure in.

Of course, the function `plug` that fills a hole has the type $(\partial F\,X)\times X\to F\,X$.

So far, the syntax ∂ denotes the differentiation of functors, i.e. of a kind of type functions with one argument. But there is also a handy expression oriented notation ∂_X slightly more suitable for calculation. The subscript indicates the variable with respect to which we want to differentiate. In general, we have

$$(\partial F)\,X = \partial_X(F\,X)$$

An example is

$$\partial(Id\times Id)\,X = \partial_X(X\times X) = 1\times X+X\times 1\cong 2\times X$$

Of course, ∂_X is just point-wise whereas ∂ is point-free style.

Exercises:

1. Rewrite some rules in point-wise style. For example, the left hand side of the product rule becomes $\partial_X(F\,X \times G\,X) = \ldots$.
2. To get familiar with one-hole contexts, differentiate the product $X^n := X \times X \times \cdots \times X$ of exactly n factors formally and convince yourself that the result is indeed the corresponding one-hole context.
3. Of course, one-hole contexts are useless if we cannot plug values of type X back into them. Write the `plug` functions corresponding to the five rules.
4. Formulate the **chain rule** for **two variables** and prove that it yields one-hole contexts. You can do this by viewing a bifunctor $F\,X\,Y$ as an normal functor in the pair (X,Y). Of course, you may need a handy notation for partial derivatives of bifunctors in point-free style.

46.2.2 Zippers via Differentiation

The above rules enable us to construct **zipper**s for recursive data types $\mu F := \mu X.F\,X$ where F is a polynomial functor. A zipper is a focus on a particular subtree, i.e. substructure of type μF inside a large tree of the same type. As in the previous chapter, it can be represented by the subtree we want to focus at and the thread, that is the context in which the subtree resides

$$\mathit{Zipper}_F = \mu F \times \mathit{Context}_F$$

.

Now, the context is a series of steps each of which chooses a particular subtree μF among those in $F\,\mu F$. Thus, the unchosen subtrees are collected together by the one-hole context $\partial F\,(\mu F)$. The hole of this context comes from removing the subtree we've chosen to enter. Putting things together, we have

$$\mathit{Context}_F = \mathit{List}\,(\partial F\,(\mu F))$$

.

or equivalently

$$\mathit{Context}_F = 1 + \partial F\,(\mu F) \times \mathit{Context}_F$$

.

To illustrate how a concrete calculation proceeds, let's systematically construct the zipper for our labyrinth data type

```
data Node a = DeadEnd a
            | Passage a (Node a)
            | Fork a (Node a) (Node a)
```

This recursive type is the fixed point

$$\mathit{Node}\,A = \mu X.\,\mathit{NodeF}_A\,X$$

of the functor

$$\mathit{NodeF}_A\,X = A + A \times X + A \times X \times X$$

.

In other words, we have

$$\mathit{Node}\,A \cong \mathit{NodeF}_A\,(\mathit{Node}\,A) \cong A + A \times \mathit{Node}\,A + A \times \mathit{Node}\,A \times \mathit{Node}\,A$$

.

The derivative reads

$$\partial_X(\mathit{NodeF}_A\,X) \cong A + 2 \times A \times X$$

and we get

$$\partial\mathit{NodeF}_A\,(\mathit{Node}\,A) \cong A + 2 \times A \times \mathit{Node}\,A$$

.

Thus, the context reads

$$\mathit{Context}_{\mathit{NodeF}} \cong \mathit{List}\,(\partial\mathit{NodeF}_A\,(\mathit{Node}\,A)) \cong \mathit{List}\,(A + 2 \times A \times (\mathit{Node}\,A))$$

.

Comparing with

```
data Branch a  = KeepStraightOn a
               | TurnLeft  a (Node a)
               | TurnRight a (Node a)
type Thread a  = [Branch a]
```

we see that both are exactly the same as expected!

Exercises:

1. Redo the zipper for a ternary tree, but with differentiation this time.
2. Construct the zipper for a list.
3. Rhetorical question concerning the previous exercise: what's the difference between a list and a stack?

46.2.3 Differentation of Fixed Point

There is more to data types than sums and products, we also have a fixed point operator with no direct correspondence in calculus. Consequently, the table is missing a rule of differentiation, namely how to differentiate fixed points $\mu F\,X = \mu Y.\,F\,X\,Y$:

$$\partial_X(\mu F\,X) = ?$$

.

As its formulation involves the chain rule in two variables, we delegate it to the exercises. Instead, we will calculate it for our concrete example type $\mathit{Node}\,A$:

$$\begin{array}{rcl} \partial_A(\mathit{Node}\,A) & = & \partial_A(A + A \times \mathit{Node}\,A + A \times \mathit{Node}\,A \times \mathit{Node}\,A) \\ & \cong & 1 + \mathit{Node}\,A + \mathit{Node}\,A \times \mathit{Node}\,A \\ & & +\partial_A(\mathit{Node}\,A) \times (A + 2 \times A \times \mathit{Node}\,A). \end{array}$$

Of course, expanding $\partial_A(\mathit{Node}\,A)$ further is of no use, but we can see this as a fixed point equation and arrive at

$$\partial_A(\mathit{Node}\,A) = \mu X.\,T\,A + S\,A \times X$$

with the abbreviations

$$T\,A = 1 + \mathit{Node}\,A + \mathit{Node}\,A \times \mathit{Node}\,A$$

and

$$S\,A = A + 2 \times A \times \mathit{Node}\,A$$

.

The recursive type is like a list with element types $S\,A$, only that the empty list is replaced by a base case of type $T\,A$. But given that the list is finite, we can replace the base case with 1 and pull $T\,A$ out of the list:

$$\partial_A(Node\,A) \cong T\,A \times (\mu X.\,1 + S\,A \times X) = T\,A \times List\,(S\,A)$$

.

Comparing with the zipper we derived in the last paragraph, we see that the list type is our context

$$List\,(S\,A) \cong Context_{NodeF}$$

and that

$$A \times T\,A \cong Node\,A$$

.

In the end, we have

$$Zipper_{NodeF} \cong \partial_A(Node\,A) \times A$$

.

Thus, differentiating our concrete example $Node\,A$ with respect to A yields the zipper up to an A!

Exercises:

1. Use the chain rule in two variables to formulate a rule for the differentiation of a fixed point.
2. Maybe you know that there are inductive (μ) and coinductive fixed points (ν). What's the rule for coinductive fixed points?

46.2.4 Differentation with respect to functions of the argument

When finding the type of a one-hole context one does d f(x)/d x. It is entirely possible to solve expressions like d f(x)/d g(x). For example, solving d x^4 / d x^2 gives 2x^2 , a two-hole context of a 4-tuple. The derivation is as follows let u=x^2 d x^4 / d x^2 = d u^2 /d u = 2u = 2 x^2 .

46.2.5 Zippers vs Contexts

In general however, zippers and one-hole contexts denote different things. The zipper is a focus on arbitrary subtrees whereas a one-hole context can only focus on the argument of a type constructor. Take for example the data type

```
data Tree a = Leaf a | Bin (Tree a) (Tree a)
```

which is the fixed point

$$\mathit{Tree}\,A = \mu X.\,A + X \times X$$

.

The zipper can focus on subtrees whose top is `Bin` or `Leaf` but the hole of one-hole context of $\mathit{Tree}\,A$ may only focus a `Leaf`s because this is where the items of type A reside. The derivative of $\mathit{Node}\,A$ only turned out to be the zipper because every top of a subtree is always decorated with an A.

Exercises:

1. Surprisingly, $\partial_A(\mathit{Tree}\,A) \times A$ and the zipper for $\mathit{Tree}\,A$ again turn out to be the same type. Doing the calculation is not difficult but can you give a reason why this has to be the case?
2. Prove that the zipper construction for μF can be obtained by introducing an auxiliary variable Y, differentiating $\mu X.\,Y \times F\,X$ with respect to it and re-substituting $Y = 1$. Why does this work?
3. Find a type $G\,A$ whose zipper is different from the one-hole context.

46.2.6 Conclusion

We close this section by asking how it may happen that rules from calculus appear in a discrete setting. Currently, nobody knows. But at least, there is a discrete notion of **linear**, namely in the sense of ”exactly once”. The key feature of the function that plugs an item of type X into the hole of a one-hole context is the fact that the item is used exactly once, i.e. linearly. We may think of the plugging map as having type

$$\partial_X F\,X \to (X \multimap F\,X)$$

where $A \multimap B$ denotes a linear function, one that does not duplicate or ignore its argument, as in linear logic. In a sense, the one-hole context is a representation of the function space $X \multimap F\,X$, which can be thought of being a linear approximation to $X \to F\,X$.

46.3 See Also

w:Zipper (data structure)[9]

- Zipper[10] on the haskell.org wiki
- Generic Zipper and its applications[11]
- Zipper-based file server/OS[12]
- Scrap Your Zippers: A Generic Zipper for Heterogeneous Types[13]

[9] https://en.wikipedia.org/wiki/Zipper%20%28data%20structure%29

[10] http://www.haskell.org/haskellwiki/Zipper

[11] http://okmij.org/ftp/Computation/Continuations.html#zipper

[12] http://okmij.org/ftp/Computation/Continuations.html#zipper-fs

[13] http://www.michaeldadams.org/papers/scrap_your_zippers/

47 Lenses and functional references

This chapter is about *functional references*. By ”references”, we mean they point at parts of values, allowing us to access and modify them. By ”functional”, we mean they do so in a way that provides the flexibility and composability we came to expect from functions. We will study functional references as implemented by the powerful `lens`[1] library. `lens` is named after *lenses*, a particularly well known kind of functional reference. Beyond being very interesting from a conceptual point of view, lenses and other functional references allow for several convenient and increasingly common idioms, put into use by a number of useful libraries.

47.1 A taste of lenses

As a warm-up, we will demonstrate the simplest use case for lenses: as a nicer alternative to the vanilla Haskell records. There will be little in the way of explanations in this section; we will fill in the gaps through the remainder of the chapter.

Consider the following types, which are not unlike something you might find in a 2D drawing library:

```
-- A point in the plane.
data Point = Point
    { positionX :: Double
    , positionY :: Double
    } deriving (Show)

-- A line segment from one point to another.
data Segment = Segment
    { segmentStart :: Point
    , segmentEnd :: Point
    } deriving (Show)

-- Helpers to create points and segments.
makePoint :: (Double, Double) -> Point
makePoint (x, y) = Point x y

makeSegment :: (Double, Double) -> (Double, Double) -> Segment
makeSegment start end = Segment (makePoint start) (makePoint end)
```

Record syntax gives us functions for accessing the fields. With them, getting the coordinates of the points that define a segment is easy enough:

```
GHCi> let testSeg = makeSegment (0, 1) (2, 4)
GHCi> positionY . segmentEnd $ testSeg
GHCi> 4.0
```

[1] https://hackage.haskell.org/package/lens

Updates, however, are clunkier...

```
GHCi> testSeg { segmentEnd = makePoint (2, 3) }
Segment {segmentStart = Point {positionX = 0.0, positionY = 1.0}
, segmentEnd = Point {positionX = 2.0, positionY = 3.0}}
```

... and get downright ugly when we need to reach a nested field. Here is what it takes to double the value of the *y* coordinate of the end point:

```
GHCi> :set +m -- Enabling multi-line input in GHCi.
GHCi> let end = segmentEnd testSeg
GHCi| in testSeg { segmentEnd = end { positionY = 2 * positionY end } }
Segment {segmentStart = Point {positionX = 0.0, positionY = 1.0}
, segmentEnd = Point {positionX = 2.0, positionY = 8.0}}
```

Lenses allow us to avoid such nastiness, so let's start over with them:

```
-- Some of the examples in this chapter require a few GHC extensions:
-- TemplateHaskell is needed for makeLenses; RankNTypes is needed for
-- a few type signatures later on.
{-# LANGUAGE TemplateHaskell, RankNTypes #-}

import Control.Lens

data Point = Point
    { _positionX :: Double
    , _positionY :: Double
    } deriving (Show)
makeLenses ''Point

data Segment = Segment
    { _segmentStart :: Point
    , _segmentEnd :: Point
    } deriving (Show)
makeLenses ''Segment

makePoint :: (Double, Double) -> Point
makePoint (x, y) = Point x y

makeSegment :: (Double, Double) -> (Double, Double) -> Segment
makeSegment start end = Segment (makePoint start) (makePoint end)
```

The only real change here is the use of `makeLenses`, which automatically generates lenses for the fields of `Point` and `Segment` (the extra underscores are required by the naming conventions of `makeLenses`). As we will see, writing lenses definitions by hand is not difficult at all; however, it can be tedious if there are lots of fields to make lenses for, and thus automatic generation is very convenient.

Thanks to `makeLenses`, we now have a lens for each field. Their names match that of the fields, except with the leading underscore removed:

```
GHCi> :info positionY
positionY :: Lens' Point Double
          -- Defined at WikibookLenses.hs:9:1
GHCi> :info segmentEnd
segmentEnd :: Lens' Segment Point
          -- Defined at WikibookLenses.hs:15:1
```

The type `positionY :: Lens' Point Double` tells us that `positionY` is a *reference* to a `Double` within a `Point`. To work with such references, we use the combinators provided by the lens library. One of them is view, which gives us the value pointed at by a lens, just like a record accessor:

```
GHCi> let testSeg = makeSegment (0, 1) (2, 4)
GHCi> view segmentEnd testSeg
Point {_positionX = 2.0, _positionY = 4.0}
```

Another one is set, which overwrites the value pointed at:

```
GHCi> set segmentEnd (makePoint (2, 3)) testSeg
Segment {_segmentStart = Point {_positionX = 0.0, _positionY = 1.0}
, _segmentEnd = Point {_positionX = 2.0, _positionY = 3.0}}
```

One of the great things about lenses is that they are easy to compose:

```
GHCi> view (segmentEnd . positionY) testSeg
GHCi> 4.0
```

Note that when writing composed lenses, such as segmentEnd . positionY, the order is from large to small. In this case, the lens that focuses on a point of the segment comes before the one that focuses on a coordinate of that point. While that might look a little surprising in contrast to how record accessors work (compare with the equivalent lens-less example at the beginning of this section), the (.) used here is just the function composition operator we know and love.

Composition of lenses provide a way out of the nested record update quagmire. Here is a translation of the coordinate-doubling example using over, through which we can apply a function to the value pointed at by a lens:

```
GHCi> over (segmentEnd . positionY) (2 *) testSeg
Segment {_segmentStart = Point {_positionX = 0.0, _positionY = 1.0}
, _segmentEnd = Point {_positionX = 2.0, _positionY = 8.0}}
```

These initial examples might look a bit magical at first. What makes it possible to use one and the same lens to get, set and modify a value? How come composing lenses with (.) just works? Is it really so easy to write lenses without the help of makeLenses? We will answer such questions by going behind the curtains to find what lenses are made of.

47.2 The scenic route to lenses

There are many ways to make sense of lenses. We will follow a sinuous yet gentle path, one which avoids conceptual leaps of faith. Along the way, we will introduce a few different kinds of functional references. Following lens terminology, from now on we will use the word *"optics"* to refer collectively to the various species of functional references. As we will see, the optics in lens are interrelated, forming a hierarchy. It is this hierarchy which we are now going to explore.

47.2.1 Traversals

We will begin not with lenses, but with a closely related optic: traversals. The Traversable[2] chapter discussed how `traverse` makes it possible to walk across a structure while producing an overall effect:

```
traverse
  :: (Applicative f, Traversable t) => (a -> f b) -> t a -> f (t b)
```

With `traverse`, you can use any `Applicative` you like to produce the effect. In particular, we have seen how `fmap` can be obtained from `traverse` simply by picking `Identity` as the applicative functor, and that the same goes for `foldMap` and `Const m`, using `Monoid m => Applicative (Const m)`:

```
fmap f = runIdentity . traverse (Identity . f)
foldMap f = getConst . traverse (Const . f)
```

`lens` takes this idea and lets it blossom.

Manipulating values within a `Traversable` structure, as `traverse` allows us to, is an example of targeting parts of a whole. As flexible as it is, however, `traverse` only handles a rather limited range of targets. For one, we might want to walk across structures that are not `Traversable` functors. Here is an entirely reasonable function that does so with our `Point` type:

```
pointCoordinates
  :: Applicative f => (Double -> f Double) -> Point -> f Point
pointCoordinates g (Point x y) = Point <$> g x <*> g y
```

`pointCoordinates` is a *traversal* of `Point`. It looks a lot like a typical implementation of `traverse`, and can be used in pretty much the same way. Here is an adaptation of the `rejectWithNegatives` example from the Traversable[3] chapter:

```
GHCi> let deleteIfNegative x =  if x < 0 then Nothing else Just x
GHCi> pointCoordinates deleteIfNegative (makePoint (1, 2))
Just (Point {_positionX = 1.0, _positionY = 2.0})
GHCi> pointCoordinates deleteIfNegative (makePoint (-1, 2))
Nothing
```

This generalised notion of a traversal that `pointCoordinates` exemplifies is captured by one of the core types of `lens`: `Traversal`.

```
type Traversal s t a b =
  forall f. Applicative f => (a -> f b) -> s -> f t
```

Note:
The `forall f.` on the right side of the `type` declaration means that any `Applicative` can be used to replace `f`. That makes it unnecessary to mention `f` on the left side, or to specify which `f` to pick when using a `Traversal`.

[2] Chapter 42 on page 269
[3] Chapter 42.1 on page 269

With the `Traversal` synonym, the type of `pointCoordinates` can be expressed as:

```
Traversal Point Point Double Double
```

Let's have a closer look at what became of each type variable in `Traversal s t a b`:

- `s` becomes `Point`: `pointCoordinates` is a traversal of a `Point`.
- `t` becomes `Point`: `pointCoordinates` produces a `Point` (in some `Applicative` context).
- `a` becomes `Double`: `pointCoordinates` targets `Double` values in a `Point` (the X and Y coordinates of the points).
- `b` becomes `Double`: the targeted `Double` values become `Double` values (possibly different than the original ones).

In the case of `pointCoordinates`, `s` is the same as `t`, and `a` is the same as b. `pointCoordinates` does not change the type of the traversed structure, or that of the targets in it, but that need not be the case. One example is good old `traverse`, whose type can be expressed as:

```
Traversable t => Traversal (t a) (t b) a b
```

`traverse` is able to change the types of the targeted values in the `Traversable` structure and, by extension, the type of the structure itself.

The Control.Lens.Traversal[4] module includes generalisations of Data.Traversable[5] functions and various other tools for working with traversals.

Exercises:

1. Write `extremityCoordinates`, a traversal that goes through all coordinates of the points that define a `Segment` in the order suggested by the `data` declaration. (Hint: use the `pointCoordinates` traversal.)

47.2.2 Setters

Next in our programme comes the generalisation of the links between `Traversable`, `Functor` and `Foldable`. We shall begin with `Functor`.

To recover `fmap` from `traverse`, we picked `Identity` as the applicative functor. That choice allowed us to modify the targeted values without producing any extra effects. We can reach similar results by picking the definition of a `Traversal`...

```
forall f. Applicative f => (a -> f b) -> s -> f t
```

... and specialising `f` to `Identity`:

```
(a -> Identity b) -> s -> Identity t
```

[4] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Traversal.html

[5] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Traversable.html

In lens parlance, that is how you get a Setter. For technical reasons, the definition of Setter in Control.Lens.Setter[6] is a little different...

```
type Setter s t a b =
  forall f. Settable f => (a -> f b) -> s -> f t
```

... but if you dig into the documentation you will find that a Settable functor is either Identity or something very much like it, so the difference need not concern us.

When we take Traversal and restrict the choice of f we actually make the type more general. Given that a Traversal works with any Applicative functor, it will also work with Identity, and therefore any Traversal is a Setter and can be used as one. The reverse, however, is not true: not all setters are traversals.

over is the essential combinator for setters. It works a lot like fmap, except that you pass a setter as its first argument in order to specify which parts of the structure you want to target:

```
GHCi> over pointCoordinates negate (makePoint (1, 2))
Point {_positionX = -1.0, _positionY = -2.0}
```

In fact, there is a Setter called mapped that allows us to recover fmap:

```
GHCi> over mapped negate [1..4]
[-1,-2,-3,-4]
GHCi> over mapped negate (Just 3)
Just (-3)
```

Another very important combinator is set, which replaces all targeted values with a constant. set setter x = over setter (const x), analogously to how (x <$) = fmap (const x):

```
GHCi> set pointCoordinates 7 (makePoint (1, 2))
Point {_positionX = 7.0, _positionY = 7.0}
```

Exercises:

1. Use over to implement...
   `scaleSegment :: Double -> Segment -> Segment`
   ... so that scaleSegment n multiplies all coordinates of a segment by x. (Hint: use your answer to the previous exercise.)
2. Implement mapped. For this exercise, you can specialise the Settable functor to Identity. (Hint: you will need Data.Functor.Identity[a].)

[a] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Functor-Identity.html

47.2.3 Folds

Having generalised the fmap-as-traversal trick, it is time to do the same with the foldMap-as-traversal one. We will use Const to go from...

[6] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Setter.html

```
forall f. Applicative f => (a -> f b) -> s -> f t
```

... to:

```
forall r. Monoid r => (a -> Const r a) -> s -> Const r s
```

Since the second parameter of `Const` is irrelevant, we replace `b` with `a` and `t` with `s` to make our life easier.

Just like we have seen for `Setter` and `Identity`, Control.Lens.Fold[7] uses something slightly more general than `Monoid r => Const r`:

```
type Fold s a =
  forall f. (Contravariant f, Applicative f) => (a -> f a) -> s -> f s
```

[7] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Lens-Fold.html

Note:
Contravariant is a type class for *contravariant functors.* The key Contravariant method is contramap...

```
contramap :: Contravariant f => (a -> b) -> f b -> f a
```

... which looks a lot like fmap, except that it, so to say, turns the function arrow around on mapping. Types parametrised over function arguments are typical examples of Contravariant. For instance, Data.Functor.Contravariant[a] defines a Predicate type for boolean tests on values of type a:

```
newtype Predicate a = Predicate { getPredicate :: a -> Bool }
```

```
GHCi> :m +Data.Functor.Contravariant
GHCi> let largerThanFour = Predicate (> 4)
GHCi> getPredicate largerThanFour 6
True
```

Predicate is a Contravariant, and so you can use contramap to modify a Predicate so that the values are adjusted in some way before being submitted to the test:

```
GHCi> getPredicate (contramap length largerThanFour) "orange"
True
```

Contravariant has laws which are analogous to the Functor ones:

```
contramap id = id
contramap (g . f) = contramap f . contramap g
```

[a] http://hackage.haskell.org/packages/archive/contravariant/latest/doc/html/Data-Functor-Contravariant.htm

Monoid r => Const r is both a Contravariant and an Applicative. Thanks to the functor and contravariant laws, anything that is both a Contravariant and a Functor is, just like Const r, a vacuous functor, with both fmap and contramap doing nothing. The additional Applicative constraint corresponds to the Monoid r; it allows us to actually perform the fold by combining the Const-like contexts created from the targets.

Every Traversal can be used as a Fold, given that a Traversal must work with any Applicative, including those that are also Contravariant. The situation parallels exactly what we have seen for Traversal and Setter.

Control.Lens.Fold offers analogues to everything in Data.Foldable[8]. Two commonly seen combinators from that module are toListOf, which produces a list of the Fold targets...

[8] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Foldable.html

```
GHCi> -- Using the solution to the exercise in the traversals subsection.
GHCi> toListOf extremityCoordinates (makeSegment (0, 1) (2, 3))
[0.0,1.0,2.0,3.0]
```

... and `preview`, which extracts the first target of a `Fold` using the `First` monoid from Data.Monoid[9].

```
GHCi> preview traverse [1..10]
Just 1
```

47.2.4 Getters

So far we have moved from `Traversal` to more general optics (`Setter` and `Fold`) by restricting the functors available for traversing. We can also go in the opposite direction, that is, making more specific optics by broadening the range of functors they have to deal with. For instance, if we take `Fold`...

```
type Fold s a =
  forall f. (Contravariant f, Applicative f) => (a -> f a) -> s -> f s
```

... and relax the `Applicative` constraint to merely `Functor`, we obtain `Getter`:

```
type Getter s a =
  forall f. (Contravariant f, Functor f) => (a -> f a) -> s -> f s
```

As `f` still has to be both `Contravariant` and `Functor`, it remains being a `Const`-like vacuous functor. Without the `Applicative` constraint, however, we can't combine results from multiple targets. The upshot is that a `Getter` always has exactly one target, unlike a `Fold` (or, for that matter, a `Setter`, or a `Traversal`) which can have any number of targets, including zero.

The essence of `Getter` can be brought to light by specialising `f` to the obvious choice, `Const r`:

```
someGetter :: (a -> Const r a) -> s -> Const r s
```

Since a `Const r whatever` value can be losslessly converted to a `r` value and back, the type above is equivalent to:

```
someGetter' :: (a -> r) -> s -> r

someGetter' k x = getConst (someGetter (Const . k) x)
someGetter g x = Const (someGetter' (getConst . g) x)
```

An `(a -> r) -> s -> r` function, however, is just an `s -> a` function in disguise (the camouflage being continuation passing style[10]):

```
someGetter'' :: s -> a
```

[9] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-Monoid.html

[10] Chapter 45 on page 303

```
someGetter'' x = someGetter' id x
someGetter' k x = k (someGetter'' x)
```

Thus we conclude that a `Getter s a` is equivalent to a `s -> a` function. From this point of view, it is only natural that it takes exactly one target to exactly one result. It is not surprising either that two basic combinators from Control.Lens.Getter[11] are `to`, which makes a `Getter` out of an arbitrary function, and `view`, which converts a `Getter` back to an arbitrary function.

```
GHCi> -- The same as fst (4, 1)
GHCi> view (to fst) (4, 1)
4
```

[11] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Lens-Getter.html

Note:
Given what we have just said about `Getter` being less general than `Fold`, it may come as a surprise that `view` can work `Fold`s and `Traversal`s as well as with `Getter`s:

```
GHCi> :m +Data.Monoid
GHCi> view traverse (fmap Sum [1..10])
Sum {getSum = 55}
GHCi> -- both traverses the components of a pair.
GHCi> view both ([1,2],[3,4,5])
[1,2,3,4,5]
```

That is possible thanks to one of the many subtleties of the type signatures of `lens`. The first argument of `view` is not exactly a `Getter`, but a `Getting`:

```
type Getting r s a = (a -> Const r a) -> s -> Const r s

view :: MonadReader s m => Getting a s a -> m a
```

`Getting` specialises the functor parameter to `Const r`, the obvious choice for `Getter`, but leaves it open whether there will be an `Applicative` instance for it (i.e. whether `r` will be a `Monoid`). Using `view` as an example, as long as `a` is a `Monoid` `Getting a s a` can be used as a `Fold`, and so `Fold`s can be used with `view` as long as the fold targets are monoidal.
Many combinators in both `Control.Lens.Getter` and `Control.Lens.Fold` are defined in terms of `Getting` rather than `Getter` or `Fold`. One advantage of using `Getting` is that the resulting type signatures tell us more about the folds that might be performed. For instance, consider `hasn't` from `Control.Lens.Fold`:

```
hasn't :: Getting All s a -> s -> Bool
```

It is a generalised test for emptiness:

```
GHCi> hasn't traverse [1..4]
False
GHCi> hasn't traverse Nothing
True
```

`Fold s a -> s -> Bool` would work just as well as a signature for `hasn't`. However, the `Getting All` in the actual signature is quite informative, in that it strongly suggests what `hasn't` does: it converts all `a` targets in `s` to the `All` monoid (more precisely, to `All False`), folds them and extracts a `Bool` from the overall `All` result.

47.2.5 Lenses at last

If we go back to `Traversal`...

```
type Traversal s t a b =
  forall f. Applicative f => (a -> f b) -> s -> f t
```

... and relax the `Applicative` constraint to `Functor`, just as we did when going from `Fold` to `Getter`...

```
type Lens s t a b =
  forall f. Functor f => (a -> f b) -> s -> f t
```

... we finally reach the `Lens` type.

What changes when moving from `Traversal` to `Lens`? As before, relaxing the `Applicative` constraint costs us the ability to traverse multiple targets. Unlike a `Traversal`, a `Lens` always focuses on a single target. As usual in such cases, there is a bright side to the restriction: with a `Lens`, we can be sure that exactly one target will be found, while with a `Traversal` we might end up with many, or none at all.

The absence of the `Applicative` constraint and the uniqueness of targets point towards another key fact about lenses: they can be used as getters. `Contravariant` plus `Functor` is a strictly more specific constraint than just `Functor`, and so `Getter` is strictly more general than `Lens`. As every `Lens` is also a `Traversal` and therefore a `Setter`, we conclude that *lenses can be used as both getters and setters.* That explains why lenses can replace record labels.

Note:
On close reading, our claim that every `Lens` can be used as a `Getter` might seem rash. Placing the types side by side...

```
type Lens s t a b =
  forall f. Functor f => (a -> f b) -> s -> f t

type Getter s a =
  forall f. (Contravariant f, Functor f) => (a -> f a) -> s -> f s
```

... shows that going from `Lens s t a b` to `Getter s a` involves making `s` equal to `t` and `a` equal to `b`. How can we be sure that is possible for any lens? An analogous issue might be raised about the relationship between `Traversal` and `Fold`. For the moment, this question will be left suspended; we will return to it in the section about optic laws.

Here is a quick demonstration of the flexibility of lenses using `_1`, a lens that focuses on the first component of a tuple:

```
GHCi> _1 (\x -> [0..x]) (4, 1) -- Traversal
[(0,1),(1,1),(2,1),(3,1),(4,1)]
GHCi> set _1 7 (4, 1) -- Setter
(7,1)
GHCi> over _1 length ("orange", 1) -- Setter, changing the types
(6,1)
GHCi> toListOf _1 (4, 1) -- Fold
[4]
GHCi> view _1 (4, 1) -- Getter
4
```

Exercises:

1. Implement the lenses for the fields of `Point` and `Segment`, that is, the ones we generated with `makeLenses` early on. (Hint: Follow the types. Once you write the signatures down you will notice that beyond `fmap` and the record labels there is not much else you can use to write them.)
2. Implement the `lens` function, which takes a getter function `s -> a` and a setter function `s -> b -> t` and produces a `Lens s t a b`. (Hint: Your implementation will be able to minimise the repetitiveness in the solution of the previous exercise.)

47.3 Composition

The optics we have seen so far fit the shape...

```
(a -> f b) -> (s -> f t)
```

... in which:

- `f` is a `Functor` of some sort;
- `s` is the type of the whole, that is, the full structure the optic works with;
- `t` is the type of what the whole becomes through the optic;
- `a` is the type of the parts, that is, the targets within `s` that the optic focuses on; and
- `b` is the type of what the parts becomes through the optic.

One key thing those optics have in common is that *they are all functions.* More specifically, they are mapping functions that turn a function acting on a part (`a -> f b`) into a function acting on the whole (`s -> f t`). Being functions, they can be composed in the usual manner. Let's have a second look at the lens composition example from the introduction:

```
GHCi> let testSeg = makeSegment (0, 1) (2, 4)
GHCi> view (segmentEnd . positionY) testSeg
GHCi> 4.0
```

An optic modifies the function it receives as argument to make it act on a larger structure. Given that `(.)` composes functions from right to left, we find that, when reading code from left to right, the components of an optic assembled with `(.)` focus on progressively smaller parts of the original structure. The conventions used by the `lens` type synonyms match this large-to-small order, with `s` and `t` coming before `a` and `b`. The table below illustrates how we can look at what an optic does either a mapping (from small to large) or as a focusing (from large to small), using `segmentEnd . positionY` as an example:

Lens	`segmentEnd`	`positionY`	`segmentEnd . positionY`

Bare type	Functor f
	=> (Point -> f Point)
	-> (Segment -
	> f Segment)
		`Functor f`
		`=> (Double -`
		`> f Double)`
		`-> (Point -`
		`> f Point)`
			`Functor f`
			`=> (Double -`
			`> f Double)`
			`-> (Segment -`
			`> f Segment)`

”Mapping” interpretation	From a function on `Point` to a function on `Segment`.	From a function on `Double` to a function on `Point`.	From a function on `Double` to a function on `Segment`.
Type with `Lens`	`Lens Segment Segment Point Point`	`Lens Point Point Double Double`	`Lens Segment Segment Double Double`
Type with `Lens'`	`Lens' Segment Point`	`Lens' Point Double`	`Lens' Segment Double`
”Focusing” interpretation	Focuses on a `Point` within a `Segment`	Focuses on a `Double` within a `Point`	Focuses on a `Double` within a `Segment`

Note:
The `Lens'` synonym is just convenient shorthand for lenses that do not change types (that is, lenses with `s` equal to `t` and `a` equal to `b`).

```
type Lens' s a = Lens s s a a
```

There are analogous `Traversal'` and `Setter'` synonyms as well.

The types behind synonyms such as `Lens` and `Traversal` only differ in which functors they allow in place of `f`. As a consequence, optics of different kinds can be freely mixed, as long as there is a type which all of them fit. Here are some examples:

```
GHCi> -- A Traversal on a Lens is a Traversal.
GHCi> (_2 . traverse) (\x -> [-x, x]) ("foo", [1,2])
[("foo",[-1,-2]),("foo",[-1,2]),("foo",[1,-2]),("foo",[1,2])]
GHCi> -- A Getter on a Lens is a Getter.
GHCi> view (positionX . to negate) (makePoint (2,4))
-2.0
GHCi> -- A Getter on a Traversal is a Fold.
GHCi> toListOf (both . to negate) (2,-3)
[-2,3]
GHCi> -- A Getter on a Setter does not exist (there is no unifying optic).
GHCi> set (mapped . to length) 3 ["orange", "apple"]

<interactive>:49:15:
    No instance for (Contravariant Identity) arising from a use of ‘to’
    In the second argument of ‘(.)’, namely ‘to length’
    In the first argument of ‘set’, namely ‘(mapped . to length)’
    In the expression: set (mapped . to length) 3 ["orange", "apple"]
```

47.4 Operators

Several `lens` combinators have infix operator synonyms, or at least operators nearly equivalent to them. Here are the correspondences for some of the combinators we have already seen:

Prefix	**Infix**
`view _1 (1,2)`	`(1,2) ^. _1`
`set _1 7 (1,2)`	`(_1 .~ 7) (1,2)`
`over _1 (2 *) (1,2)`	`(_1 %~ (2 *)) (1,2)`
`toListOf traverse [1..4]`	`[1..4] ^.. traverse`
`preview traverse []`	`[] ^? traverse`

lens operators that extract values (e.g. (^.), (^..) and (^?)) are flipped with respect to the corresponding prefix combinators, so that they take the structure from which the result is extracted as the first argument. That improves readability of code using them, as writing the full structure before the optics targeting parts of it mirrors how composed optics are written in large-to-small order. With the help of the (&) operator, which is defined simply as flip ($), the structure can also be written first when using modifying operators (e.g. (.~) and (%~)). (&) is particularly convenient when there are many fields to modify:

```
sextupleTest = (0,1,0,1,0,1)
    & _1 .~ 7
    & _2 %~ (5 *)
    & _3 .~ (-1)
    & _4 .~ "orange"
    & _5 %~ (2 +)
    & _6 %~ (3 *)
```

```
GHCi> sextupleTest
(7,5,-1,"orange",2,3)
```

47.5 A swiss army knife

Thus far we have covered enough of lens to introduce lenses and show that they aren't arcane magic. That, however, is only the tip of the iceberg. lens is a large library providing a rich assortment of tools, which in turn realise a colourful palette of concepts. The odds are that if you think of anything in the core libraries there will be a combinator somewhere in lens that works with it. It is no exaggeration to say that a book exploring every corner of lens might be made as long as this one you are reading. Unfortunately, we cannot undertake such an endeavour right here. What we can do is briefly discussing a few other general-purpose lens tools you are bound to encounter in the wild at some point.

47.5.1 State manipulation

There are quite a few combinators for working with state functors peppered over the lens modules. For instance:

- use from Control.Lens.Getter is an analogue of gets from Control.Monad.State that takes a getter instead of a plain function.
- Control.Lens.Setter includes suggestive-looking operators that modify parts of a state targeted a setter (e.g. .= is analogous to set, %= to over and (+= x) to over (+x)).
- Control.Lens.Zoom[12] offers the remarkably handy zoom combinator, which uses a traversal (or a lens) to zoom into a part of a state. It does so by lifiting a stateful computation into one that works with a larger state, of which the original state is a part.

Such combinators can be used to write highly intention-revealing code that transparently manipulates deep parts of a state:

```
import Control.Monad.State
```

[12] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Zoom.html

```
stateExample :: State Segment ()
stateExample = do
    segmentStart .= makePoint (0,0)
    zoom segmentEnd $ do
        positionX += 1
        positionY *= 2
        pointCoordinates %= negate
```

```
GHCi> execState stateExample (makeSegment (1,2) (5,3))
Segment {_segmentStart = Point {_positionX = 0.0, _positionY = 0.0}
, _segmentEnd = Point {_positionX = -6.0, _positionY = -6.0}}
```

47.5.2 Isos

In our series of `Point` and `Segment` examples, we have been using the `makePoint` function as a convenient way to make a `Point` out of `(Double, Double)` pair.

```
makePoint :: (Double, Double) -> Point
makePoint (x, y) = Point x y
```

The X and Y coordinates of the resulting `Point` correspond exactly to the two components of the original pair. That being so, we can define an `unmakePoint` function...

```
unmakePoint :: Point -> (Double, Double)
unmakePoint (Point x y) = (x,y)
```

... so that `makePoint` and `unmakePoint` are a pair of *inverses*, that is, they undo each other:

```
unmakePoint . makePoint = id
makePoint . unmakePoint = id
```

In other words, `makePoint` and `unmakePoint` provide a way to losslessly convert a pair to a point and vice-versa. Using jargon, we can say that `makePoint` and `unmakePoint` form an *isomorphism.*

`unmakePoint` might be made into a `Lens' Point (Double, Double).` Symmetrically. `makePoint` would give rise to a `Lens' (Double, Double) Point`, and the two lenses would be a pair of inverses. Lenses with inverses have a type synonym of their own, `Iso`, as well as some extra tools defined in Control.Lens.Iso[13].

An `Iso` can be built from a pair of inverses through the `iso` function:

```
iso :: (s -> a) -> (b -> t) -> Iso s t a b
```

```
pointPair :: Iso' Point (Double, Double)
pointPair = iso unmakePoint makePoint
```

`Iso`s are `Lens`es, and so the familiar lens combinators work as usual:

```
GHCi> import Data.Tuple (swap)
GHCi> let testPoint = makePoint (2,3)
GHCi> view pointPair testPoint -- Equivalent to unmakePoint
```

[13] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Iso.html

```
(2.0,3.0)
GHCi> view (pointPair . _2) testPoint
3.0
GHCi> over pointPair swap testPoint
Point {_positionX = 3.0, _positionY = 2.0}
```

Additionally, `Iso`s can be inverted using `from`:

```
GHCi> :info from pointPair
from :: AnIso s t a b -> Iso b a t s
          -- Defined in ‘Control.Lens.Iso’
pointPair :: Iso' Point (Double, Double)
          -- Defined at WikibookLenses.hs:77:1
GHCi> view (from pointPair) (2,3) -- Equivalent to makePoint
Point {_positionX = 2.0, _positionY = 3.0}
GHCi> view (from pointPair . positionY) (2,3)
3.0
```

Another interesting combinator is `under`. As the name suggests, it is just like `over`, except that it uses the inverted `Iso` that `from` would give us. We will demonstrate it by using the `enum` isomorphism to play with the `Int` representation of `Char`s without using `chr` and `ord` from `Data.Char` explicitly:

```
GHCi> :info enum
enum :: Enum a => Iso' Int a          -- Defined in ‘Control.Lens.Iso’
GHCi> under enum (+7) 'a'
'h'
```

`newtype`s and other single-constructor types give rise to isomorphisms. Control.Lens.Wrapped[14] exploits that fact to provide `Iso`-based tools which, for instance, make it unnecessary to remember record label names for unwrapping `newtype`s...

```
GHCi> let testConst = Const "foo"
GHCi> -- getConst testConst
GHCi> op Const testConst
"foo"
GHCi> let testIdent = Identity "bar"
GHCi> -- runIdentity testIdent
GHCi> op Identity testIdent
"bar"
```

... and that make `newtype` wrapping for instance selection less messy:

```
GHCi> :m +Data.Monoid
GHCi> -- getSum (foldMap Sum [1..10])
GHCi> ala Sum foldMap [1..10]
55
GHCi> -- getProduct (foldMap Product [1..10])
GHCi> ala Product foldMap [1..10]
3628800
```

47.5.3 Prisms

With `Iso`, we have reached for the first time a rank below `Lens` in the hierarchy of optics: every `Iso` is a `Lens`, but not every `Lens` is an `Iso`. By going back to `Traversal`, we can observe how the optics get progressively less precise in what they point to:

[14] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Wrapped.html

- An Iso is an optic that has exactly one target and is invertible.
- A Lens also has exactly one target but is not invertible.
- A Traversable can have any number of targets and is not invertible.

Along the way, we first dropped invertibility and then the uniqueness of targets. If we follow a different path by dropping uniqueness before invertibility, we find a second kind of optic between isomorphisms and traversals: *prisms*. A Prism is an invertible optic that need not have exactly one target. As invertibility is incompatible with multiple targets, we can be more precise: a Prism can reach either no targets or exactly one target.

Aiming at a single target with the possibility of failure sounds a lot like pattern matching, and prisms are indeed able to capture that. If tuples and records provide natural examples of lenses, Maybe, Either and other types with multiple constructors play the same role for prisms.

Every Prism is a Traversal, and so the usual combinators for traversals, setters and folds all work with prisms:

```
GHCi> set _Just 5 (Just "orange")
Just 5
GHCi> set _Just 5 Nothing
Nothing
GHCi> over _Right (2 *) (Right 5)
Right 10
GHCi> over _Right (2 *) (Left 5)
Left 5
GHCi> toListOf _Left (Left 5)
[5]
```

A Prism is not a Getter, though: the target might not be there. For that reason, we use preview rather than view to retrieve the target:

```
GHCi> preview _Right (Right 5)
Just 5
GHCi> preview _Right (Left 5)
Nothing
```

For inverting a Prism, we use re and review from Control.Lens.Review[15]. re is analogous to from, though it gives merely a Getter. review is equivalent to view with the inverted prism.

```
GHCi> view (re _Right) 3
Right 3
GHCi> review _Right 3
Right 3
```

Just like there is more to lenses than reaching record fields, prisms are not limited to matching constructors. For instance, Control.Lens.Prism[16] defines only, which encodes equality tests as a Prism:

```
GHCi> :info only
only :: Eq a => a -> Prism' a ()
        -- Defined in ‘Control.Lens.Prism’
```

[15] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Review.html

[16] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Control-Lens-Prism.html

```
GHCi> preview (only 4) (2 + 2)
Just ()
GHCi> preview (only 5) (2 + 2)
Nothing
```

The `prism` and `prism'` functions allow us to build our own prisms. Here is an example using `stripPrefix` from `Data.List`:

```
GHCi> :info prism
prism :: (b -> t) -> (s -> Either t a) -> Prism s t a b
        -- Defined in ‘Control.Lens.Prism’
GHCi> :info prism'
prism' :: (b -> s) -> (s -> Maybe a) -> Prism s s a b
        -- Defined in ‘Control.Lens.Prism’
GHCi> import Data.List (stripPrefix)
GHCi> :t stripPrefix
stripPrefix :: Eq a => [a] -> [a] -> Maybe [a]
```

```
prefixed :: Eq a => [a] -> Prism' [a] [a]
prefixed prefix = prism' (prefix ++) (stripPrefix prefix)
```

```
GHCi> preview (prefixed "tele") "telescope"
Just "scope"
GHCi> preview (prefixed "tele") "orange"
Nothing
GHCi> review (prefixed "tele") "graph"
"telegraph"
```

`prefixed` is available from `lens`, in the Data.List.Lens[17] module.

Exercises:

1. `Control.Lens.Prism` defines an `outside` function, which has the following (simplified) type:
```
outside :: Prism s t a b
      -> Lens (t -> r) (s -> r) (b -> r) (a -> r)
```
   a) Explain what `outside` does without mentioning its implementation. (Hint: The documentation says that with it we can ”use a `Prism` as a kind of first-class pattern”. Your answer should expand on that, explaining how we can use it in such a way.)
   b) Use `outside` to implement `maybe` and `either` from the Prelude: `maybe :: b -> (a -> b) -> Maybe a -> beither :: (a -> c) -> (b -> c) -> Either a b -> c`

47.6 Laws

There are laws specifying how sensible optics should behave. We will now survey those that apply to the optics that we covered here.

Starting from the top of the taxonomy, `Fold` does not have laws, just like the `Foldable` class. `Getter` does not have laws either, which is not surprising, given that any function can be made into a `Getter` via `to`.

[17] http://hackage.haskell.org/packages/archive/lens/latest/doc/html/Data-List-Lens.html

Setter, however, does have laws. over is a generalisation of fmap, and is therefore subject to the functor laws:

```
over s id = id
over s g . over s f = over s (g . f)
```

As set s x = over s (const x), a consequence of the second functor law is that:

```
set s y . set s x = set s y
```

That is, setting twice is the same as setting once.

Traversal laws, similarly, are generalisations of the Traversable laws:

```
t pure = pure
fmap (t g) . t f = getCompose . t (Compose . fmap g . f)
```

The consequences discussed in the Traversable[18] chapter follow as well: a traversal visits all of its targets exactly once, and must either preserve the surrounding structure or destroy it wholly.

Every Lens is a Traversal and a Setter, and so the laws above also hold for lenses. In addition, every Lens is also a Getter. Given that a lens is both a getter and a setter, it should get the same target that it sets. This common sense requirement is expressed by the following laws:

```
view l (set l x) = x
set l (view l z) z = z
```

Together with the "setting twice" law of setters presented above, those laws are commonly referred to as the *lens laws*.

Analogous laws hold for Prisms, with preview instead of view and review instead of set:

```
preview p (review p x) = Just x
review p <$> preview p z = Just z
```

Isos are both lenses and prisms, so all of the laws above hold for them. The prism laws, however, can be simplified, given that for isomorphisms preview i = Just . view i (that is, preview never fails):

```
view i (review i x) = x
review i (view i z) = z
```

47.6.1 Polymorphic updates

When we look at optic types such as Setter s t a b and Lens s t a b we see four independent type variables. However, if we take the various optic laws into account we find out that not all choices of s, t, a and b are reasonable. For instance, consider the "setting twice" law of setters:

[18] Chapter 42.3 on page 273

```
set s y . set s x = set s y
```

For ”setting twice is the same than setting once” to make sense, it must be possible to set twice using the same setter. As a consequence, the law can only hold for a `Setter s t a b` if `t` can somehow be specialised so that it becomes equal to `s` (otherwise the type of the whole would change on every `set`, leading to a type mismatch).

From considerations about the types involved in the laws such as the one above, it follows that the four type parameters in law-abiding `Setter`s, `Traversal`s, `Prism`s and `Lens`es are not fully independent from each other. We won't examine the interdependency in detail, but merely point out some of its consequences. Firstly, `a` and `b` are cut from the same cloth, in that even if an optic can change types there must be a way of specialising `a` and `b` to make them equal; furthermore, the same holds for `s` and `t`. Secondly, if `a` and `b` are equal then `s` and `t` must be equal as well.

In practice, those restrictions mean that valid optics that can change types usually have `s` and `t` parametrised in terms of `a` and `b`. Type-changing updates in this fashion are often referred to as *polymorphic updates*. For the sake of illustration, here are a few arbitrary examples taken from `lens`:

```
-- To avoid distracting details,
-- we specialised the types of argument and _1.
mapped :: Functor f => Setter (f a) (f b) a b
contramapped :: Contravariant f => Setter (f b) (f a) a b
argument :: Setter (b -> r) (a -> r) a b
traverse :: Traversable t => Traversal (t a) (t b) a b
both :: Bitraversable r => Traversal (r a a) (r b b) a b
_1 :: Lens (a, c) (b, c) a b
_Just :: Prism (Maybe a) (Maybe b) a b
```

At this point, we can return to the question left open when we presented the `Lens` type. Given that `Lens` and `Traversal` allow type changing while `Getter` and `Fold` do not, it would be indeed rash to say that every `Lens` is a `Getter`, or that every `Traversal` is a `Fold`. However, the interdependence of the type variables mean that every *lawful* `Lens` can be used as a `Getter`, and every lawful `Traversal` can be used as a `Fold`, as lawful lenses and traversals can always be used in non type-changing ways.

47.7 No strings attached

As we have seen, we can use `lens` to define optics through functions such as `lens` and auto-generation tools such as `makeLenses`. Strictly speaking, though, these are merely convenience helpers. Given that `Lens`, `Traversal` and so forth are just type synonyms, their definitions are not needed when writing optics – for instance, we can always write `Functor f => (a -> f b) -> (s -> f t)` instead of `Lens s t a b`. That means we can define optics compatible with `lens` without using `lens` at all! In fact, any `Lens`, `Traversal`, `Setter` or `Getting` can be defined with no dependencies other than the `base` package.

The ability to define optics without depending on the `lens` library provides considerable flexibility in how they can be leveraged. While there are libraries that do depend on `lens`, library authors are often wary of acquiring a dependency on large packages with several dependencies such as `lens`, especially when writing small, general-purpose libraries. Such concerns can be sidestepped by defining the optics without using the type synonyms or the helper tools in `lens`. Furthermore, the types being only synonyms makes it possible to have multiple optic frameworks (i.e. `lens` and similar libraries) that can be used interchangeably.

47.8 Further reading

- Several paragraphs above, we said that `lens` easily provides enough material for a full book. The closest thing to that we currently have is Artyom Kazak's ”lens over tea”[19] series of blog posts. It explores the implementation of functional references in `lens` and the concepts behind it in far more depth than what we are able to do here. Highly recommended reading.
- Useful information can be reached through `lens`' GitHub wiki[20], and of course `lens`' API documentation[21] is well worth exploring.
- `lens` is a large and complex library. If you want to study its implementation but would rather begin with something simpler, a good place to start are minimalistic `lens`-compatible libraries such as `microlens`[22] and `lens-simple`[23].
- Studying (and using!) optic-powered libraries is a good way to get the hang of how functional references are used. Some arbitrary examples:
  - `diagrams`[24], a vector graphics library that uses `lens` extensively to deal with properties of graphic elements.
  - `wreq`[25], a web client library with a `lens`-based interface.
  - `xml-lens`[26], which provides optics for manipulating XML.
  - `formattable`[27], a library for date, time and number formattting. Formattable.NumFormat[28] is an example of a module that provides `lens`-compatible lenses without depending on the `lens` package.

[19] http://artyom.me/lens-over-tea-1

[20] https://github.com/ekmett/lens/wiki

[21] https://hackage.haskell.org/package/lens

[22] http://hackage.haskell.org/package/microlens

[23] http://hackage.haskell.org/package/lens-simple

[24] http://projects.haskell.org/diagrams/

[25] http://www.serpentine.com/wreq/

[26] https://hackage.haskell.org/package/xml-lens

[27] http://hackage.haskell.org/package/formattable

[28] http://hackage.haskell.org/packages/archive/formattable/latest/doc/html/Formattable-NumFormat.html

48 Mutable objects

Functional purity is a defining characteristics of Haskell, one which leads to many of its strengths. As such, language and ecosystem encourage eschewing mutable state altogether. Thanks to tools such as the `State` monad[1], which allows us to keep track of state in a convenient and functionally pure way, and efficient immutable data structures[2] like the ones provided by the `containers` and `unordered-containers` packages, Haskell programmers can get by perfectly fine with complete immutability in the vast majority of situations. However, under select circumstances using mutable state is just the most sensible option. One might, for instance, be interested in:

- From Haskell code, using a library written in another language which assumes mutable state everywhere. This situation often arises with event-callback GUI toolkits.
- Using Haskell to implement a language that provides imperative-style mutable variables.
- Implementing algorithms that inherently require destructive updates to variables.
- Dealing with volumes of bulk data massive enough to justify squeezing every drop of computational power available to make the problem at hand feasible.

Any general-purpose programming language worth its salt should be able to deal with such tasks. With Haskell, it is no different: there are not only ways to create mutable objects, but also to keep mutability under control, existing peacefully in a setting where immutability is the default.

48.1 `IORef`s

Let's begin with the simplest of those use cases above. A common way of structuring code for user interfaces is through the event-and-callback model. The event might be a button click or a key press, while the callback is just a piece of code meant to be called in response to the event. The client code (that is, your code, if you are using such a library) should set up the wiring that connects interface elements, events involving them, and the corresponding callbacks. An hypothetical function to arrange a callback might have the following type:

```
register :: (Element -> Event) -> Element -> IO () -> IO ()
```

The `IO ()` argument is the callback, while the result of `register` is an `IO` action which sets up the wiring. Running `register click button1 (print "Hello")` would lead to ”Hello” being printed on the console following every click on `button1`.

[1] Chapter 35 on page 207

[2] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FData%20structures%20primer

Both `register` – with pervasive `IO` and lacking useful return values – and our exposition above have a marked imperative feel. That's because our hypothetical GUI library was written using a more imperative style in a wholly different language. Some good soul has written a facade so that we can use it from Haskell, but the facade is a very thin one, and so the style of the original library leaks into our code [3].

Using `register` to perform `IO` actions such as printing to the console or showing dialog boxes is easy enough. However, what if we want to add 1 to a counter every time a button is clicked? The type of `register` doesn't reveal any way to pass information to the callback, nor to get information back from it (the return types are `()`). `State` doesn't help: even if there was a way to pass an initial state to the callback, run a `State` computation within it, what would we do with the results? We would need to pass the resulting state of the counter to the callback on the next time the button is clicked, and we would have no idea when that would happen, nor a place to keep the value in the meantime.

The obvious solution to this issue in many languages would be creating a *mutable* variable outside of the callback and then giving the callback a *reference* to it, so that its code can change the value of the variable at will. We need not worry, though, as Haskell allows us to do exactly that. In fact, there are several types of mutable variables available, the simplest of which is the `IORef.IORef`s are very simple; they are just boxes containing mutable values. We can create one as follows:

```
GHCi> import Data.IORef
GHCi> :t newIORef
newIORef :: a -> IO (IORef a)
GHCi> box <- newIORef (4 :: Int)
```

`newIORef` takes a value and gives back, as the result of an `IO` action, an `IORef` initialised to that value. We can then use `readIORef` to retrieve the value in it...

```
GHCi> :t readIORef
readIORef :: IORef a -> IO a
GHCi> readIORef box >>= print
4
```

... and `modifyIORef` and `writeIORef` to change it:

```
GHCi> modifyIORef box (2*)
GHCi> readIORef box >>= print
8
GHCi> writeIORef box 0
GHCi> readIORef box >>= print
0
```

An `IORef` would be enough for implementing the counter, given that it would persist between button clicks. The code might look like this:

```
setupGUI :: IORef Int -> IO ()
setupGUI counter = do
```

[3] The technical term for facades over libraries from other languages is *bindings*. Bindings can be thin, exposing transparently the constructs of the original library, or they can add extra layers of abstraction can be built on to achieve a more Haskell-like feel. The elementary tool for creating bindings in Haskell is the *foreign function interface*, which we cover in a chapter of Haskell in Practice ˆ{Chapter82 on page 535}.

```
    -- However much other GUI preparation code we need.
    register click button1 (modifyIORef counter (+1))

main :: IO ()
main = do
    -- etc.
    counter <- newIORef (0 :: Int)
    setupGUI counter
    -- Then just use the counter value wherever necessary.
```

Note there is no point in using `IORef`s indiscriminately, without a good reason for it. Beyond the more fundamental concerns with mutable state, it just would not be very convenient to do so with all those explicit read/write/modify calls, not to mention the need to introduce `IO` in extra places to handle the `IORef` (in our hypothetical example that wouldn't be much of an issue, as the GUI code would have to live in `IO` anyway, and we presumably would keep it apart from the pure functions forming the core of our program, as good Haskell practice dictates). Still, `IORef`s are there for when you can't avoid them.

48.1.1 The pitfalls of concurrency

There is another very important use case for mutable variables that we didn't mention in the introduction: *concurrency*, that is, circumstances when simultaneous computations are being executed by the computer. Concurrency scenarios range from the trivial (a progress bar displaying the status of a background task) to the extremely complex (server-side software handling thousands of requests at once). Given that in principle nothing guarantees that simultaneous computations will run in step with each other, any communication between them calls for mutable variables. That, however, introduces a complication: the issues with understandability and predictability of code using mutable state become much more serious in the presence of independent computations with unpredictable timings. For instance, computation A might need the result of computation B, but it might ask for that result earlier than predicted and thus acquire a bogus result. Writing correct concurrent code can be difficult, and subtle bugs are easy to introduce unless adequate measures are taken.

The only functions in Data.IORef[4] that provide extra safety in concurrent code are `atomicallyModifyIORef`, `atomicallyModifyIORef'` and `atomicallyWriteIORef`, which are only of any help in very simple situations in which there is just one `IORef` meant to be used as a shared resource between computations. Concurrent Haskell code should take advantage of more sophisticated tools tailored for concurrency, such as `MVar`s (mutable variables that a computation can make unavailable to the others for as long as necessary – see Control.Concurrent.MVar[5]) and Control.Concurrent.STM[6] from the `stm` package (an implementation of *software transactional memory*, a concurrency model which makes it possible to write safe concurrent code while avoiding the ugliness and complications of having to explicitly manage the availability of all shared variables) [7].

[4] `http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-IORef.html`

[5] `http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Concurrent-MVar.html`

[6] `http://hackage.haskell.org/packages/archive/stm/latest/doc/html/Control-Concurrent-STM.html`

[7] A future chapter of this book will introduce some of those features.

48.2 The ST monad

In the IORef example above, mutability was imposed upon our code by external demands. However, in the two final scenarios suggested by the introduction (algorithms that require mutability and extreme computational demands) the need for mutable state is internal – that is, it is not reflected in any way in the overall results. For instance, sorting a list does not require mutability in any essential way, and so a function that sorts a list and returns a new lists should, in principle, be functionally pure even if the sorting algorithm uses destructive updates to swap the position of the elements. In such case, the mutability is just an implementation detail. The standard libraries provide a nifty tool for handling such situations while still ending up with pure functions: the ST monad, which can be found in Control.Monad.ST[8].

```
data ST s a
```

ST s a looks a lot like State s a, and indeed they are similar in spirit. An ST computation is one that uses an internal state to produce results, except that the state is mutable. For that purpose, Data.STRef[9] provides STRefs. A STRef s a is exactly like an IORef a, but it lives in the ST s monad rather than in IO.

There is one *major* difference that sets apart ST from both State and IO. Control.Monad.ST offers a runST function with the following type:

```
runST :: (forall s. ST s a) -> a
```

At first, that is a shocking type signature. If ST involves mutability, how come we can simply extract a values from the monad? The answer lies in the forall s. part of the type. Having a forall s. enclosed within the type of an argument amounts to telling the type checker ”s could be anything. Don't make any assumptions about it”. Not making any assumptions, however, means that s cannot be matched with anything else – even with the s from another invocation of runST [10]:

```
GHCi> import Control.Monad.ST
GHCi> import Data.STRef
GHCi> -- Attempt to keep an STRef around to pass to pure code:
GHCi> let ref = runST $ newSTRef (4 :: Int)

<interactive>:125:19:
    Couldn't match type ‘a’ with ‘STRef s Int’
      because type variable ‘s’ would escape its scope
    This (rigid, skolem) type variable is bound by
      a type expected by the context: ST s a
      at <interactive>:125:11-37
    Expected type: ST s a
      Actual type: ST s (STRef s Int)
    Relevant bindings include ref :: a (bound at <interactive>:125:5)
    In the second argument of ‘($)’, namely ‘newSTRef (4 :: Int)’
    In the expression: runST $ newSTRef (4 :: Int)
GHCi> -- The error message is quite clear:
GHCi> -- "because type variable ‘s’ would escape its scope"
```

[8] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Control-Monad-ST.html

[9] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Data-STRef.html

[10] This is an example of an *existential type.* ”Existential” is meant in a precise technical sense, but we can get the gist of it by noting that the only thing we know about it is that it exists.

```
GHCi> -- Attempt to feed an STRef from one ST computation to another:
GHCi> let x = runST $ readSTRef =<< runST (newSTRef (4 :: Int))

<interactive>:129:38:
    Couldn't match type ‘STRef s1 Int’ with ‘ST s (STRef s a)’
    Expected type: ST s1 (ST s (STRef s a))
      Actual type: ST s1 (STRef s1 Int)
    Relevant bindings include x :: a (bound at <interactive>:129:5)
    In the first argument of ‘runST’, namely ‘(newSTRef (4 :: Int))’
    In the second argument of ‘(=<<)’, namely
      ‘runST (newSTRef (4 :: Int))’
GHCi> -- The 's' from each computation are necessarily not the same.
```

The overall effect of this type trickery is to insulate the internal state and mutability within each ST computation, so that from the point of view of anything else in the program runST is a pure function.

As a trivial example of ST in action, here is a very imperative-looking version of sum for lists [11]:

```
import Control.Monad.ST
import Data.STRef
import Data.Foldable

sumST :: Num a => [a] -> a
sumST xs = runST $ do
    n <- newSTRef 0
    for_ xs $ \x ->
        modifySTRef n (+x)
    readSTRef n
```

For all intents and purposes, sumST is no less pure than the familiar sum. The fact that it destructively updates its accumulator n is a mere implementation detail, and there is no way information about n could leak other than through the final result. Looking at a simple example like this one makes it clear that the s type variable in ST s a does not correspond to anything in particular within the computation – it is just an artificial marker. Another detail worth noting is that even though for_ folds the list from the right the sums are done from the left, as the mutations are performed as applicative effects sequenced form left to right.

48.3 Mutable data structures

Mutable data structures can be found in the libraries for the exceptional use cases for which they prove necessary. For instance, mutable arrays (alongside with immutable ones) can be found in the vector[12] package or the array[13] package bundled with GHC [14]. There are also mutable hash tables, such as those from the hashtables package[15]. In all cases mentioned, both ST and IO versions are provided.

[11] Adapted from the HaskellWiki page on ST ^{https://wiki.haskell.org/Monad/ST} .

[12] https://hackage.haskell.org/package/vector

[13] https://hackage.haskell.org/package/array

[14] For general observations on arrays, see the data structures primer ^{https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FData%20structures%20primer%23Raw%20performance%20with%20arrays} .

[15] https://hackage.haskell.org/package/hashtables

48.4 Further reading

- The seventh chapter of Write Yourself a Scheme in 48 Hours[16] provides an interesting example of `IORefs` being used to implement mutable variables in a language.
- Lennart Augustsson's blog[17] shows how a true quicksort (that is, one using the original algorithm which performs destructive updates to sort the list) can be implemented in Haskell, just like we assured that was possible way back in Haskell/Higher-order functions[18]. His implementation is quite amusing thanks to the combinators used to handle mutability, which make Haskell look like C. Be sure to check the two posts before the one linked to see how that was done.

[16] https://en.wikibooks.org/wiki/Write_Yourself_a_Scheme_in_48_Hours%2FAdding_Variables_and_Assignment

[17] http://augustss.blogspot.com.br/2007/08/quicksort-in-haskell-quicksort-is.html

[18] Chapter 19 on page 117

49 Concurrency

49.1 Concurrency

Concurrency in Haskell is mostly done with Haskell threads. Haskell threads are user-space threads that are implemented in the runtime. Haskell threads are much more efficient in terms of both time and space than Operating System threads. Apart from traditional synchronization primitives like semaphores, Haskell offers Software Transactional Memory which greatly simplifies concurrent access to shared memory.

The modules for concurrency are Control.Concurrent.* and Control.Monad.STM.

49.2 When do you need it?

Perhaps more important than **when** is **when not**. Concurrency in Haskell is not used to utilize multiple processor cores; you need another thing, ”parallelism[1]”, for that. Instead, concurrency is used for when a single core must divide its attention between various things, typically IO.

For example, consider a simple ”static” webserver (i.e. serves only static content such as images). Ideally, such a webserver should consume few processing resources; instead, it must be able to transfer data as fast as possible. The bottleneck should be I/O, where you can throw more hardware at the problem. So you must be able to efficiently utilize a single processor core among several connections.

In a C version of such a webserver, you'd use a big loop centered around `select()` on each connection and on the listening socket. Each open connection would have an attached data structure specifying the state of that connection (i.e. receiving the HTTP header, parsing it, sending the file). Such a big loop would be difficult and error-prone to code by hand. However, using Concurrent Haskell, you would be able to write a much smaller loop concentrating solely on the listening socket, which would spawn a new ”thread” for each accepted connection. You can then write a new ”thread” in the IO monad which, in sequence, receives the HTTP header, parses it, and sends the file.

Internally, the Haskell compiler will then convert the spawning of the thread to an allocation of a small structure specifying the state of the ”thread”, congruent to the data structure you would have defined in C. It will then convert the various threads into a single big loop. Thus, while you write as if each thread is independent, internally the compiler will convert it to a big loop centered around `select()` or whatever alternative is best on your system.

[1] https://en.wikibooks.org/wiki/Haskell%2FParallelism

49.3 Example

Example: Downloading files in parallel

```
downloadFile :: URL -> IO ()
downloadFile = undefined

downloadFiles :: [URL] -> IO ()
downloadFiles = mapM_ (forkIO . downloadFile)
```

49.4 Software Transactional Memory

Software Transactional Memory (STM) is a mechanism that allows transactions on memory similar to database transactions. It greatly simplifies access to shared resources when programming in a multithreaded environment. By using STM, you no longer have to rely on locking.

To use STM, you have to include Control.Monad.STM. To change into the STM-Monad the atomically function is used. STM offers different primitives (TVar, TMVar, TChan and TArray) that can be used for communication.

The following example shows how to use a TChan to communicate between two threads. The channel is created in the main function and handed over to the reader/writerThread functions. The readerThread waits on the TChan for new input and prints it. The writerThread writes some Int-values to the channel and terminates.

Example: Communication with a TChan

```
import Control.Monad.STM
import Control.Concurrent
import Control.Concurrent.STM.TChan

oneSecond = 1000000

writerThread :: TChan Int -> IO ()
writerThread chan = do
        atomically $ writeTChan chan 1
        threadDelay oneSecond
        atomically $ writeTChan chan 2
        threadDelay oneSecond
        atomically $ writeTChan chan 3
        threadDelay oneSecond

readerThread :: TChan Int -> IO ()
readerThread chan = do
        newInt <- atomically $ readTChan chan
        putStrLn $ "read new value: " ++ show newInt
        readerThread chan

main = do
        chan <- atomically $ newTChan
        forkIO $ readerThread chan
        forkIO $ writerThread chan
        threadDelay $ 5 * oneSecond
```

50 Fun with Types

51 Polymorphism basics

51.1 Parametric Polymorphism

Section goal = short, enables reader to read code (ParseP) with ∀and use libraries (ST) without horror. Question Talk:Haskell/The_Curry-Howard_isomorphism#Polymorphic types[1] would be solved by this section.

Link to the following paper: Luca Cardelli: On Understanding Types, Data Abstraction, and Polymorphism[2].

51.1.1 forall a

As you may know, a **polymorphic** function is a function that works for many different types. For instance,

```
length :: [a] -> Int
```

can calculate the length of any list, be it a string `String = [Char]` or a list of integers `[Int]`. The **type variable** `a` indicates that `length` accepts any element type. Other examples of polymorphic functions are

```
fst :: (a, b) -> a
snd :: (a, b) -> b
map :: (a -> b) -> [a] -> [b]
```

Type variables always **begin in lowercase** whereas concrete types like `Int` or `String` always start with an uppercase letter, that's how we can tell them apart.

There is a more **explicit** way to indicate that `a` can be any type

```
length :: forall a. [a] -> Int
```

In other words, "for all types `a`, the function `length` takes a list of elements of type `a` and returns an integer". You should think of the old signature as an abbreviation for the new one with the `forall`[3]. That is, the compiler will internally insert any missing `forall` for you. Another example: the types signature for `fst` is really a shorthand for

```
fst :: forall a. forall b. (a,b) -> a
```

[1] https://en.wikibooks.org/wiki/Talk%3AHaskell%2FThe_Curry-Howard_isomorphism%23Polymorphic%20types

[2] http://lucacardelli.name/Papers/OnUnderstanding.A4.pdf

[3] Note that the keyword `forall` is not part of the Haskell 98 standard, but any of the language extensions `ScopedTypeVariables`, `Rank2Types` or `RankNTypes` will enable it in the compiler. A future Haskell standard will incorporate one of these.

or equivalently

```
fst :: forall a b. (a,b) -> a
```

Similarly, the type of `map` is really

```
map :: forall a b. (a -> b) -> [a] -> [b]
```

The idea that something is applicable to every type or holds for everything is called **universal quantification**[4]. In mathematical logic, the symbol ∀[5] (an upside-down A, read as "forall") is commonly used for that, it is called the **universal quantifier**.

51.1.2 Higher rank types

With explicit `forall`, it now becomes possible to write functions that expect **polymorphic arguments**, like for instance

```
foo :: (forall a. a -> a) -> (Char,Bool)
foo f = (f 'c', f True)
```

Here, `f` is a polymorphic function, it can be applied to anything. In particular, `foo` can apply it to both the character `'c'` and the boolean `True`.

It is not possible to write a function like `foo` in Haskell98, the type checker will complain that `f` may only be applied to values of either the type `Char` or the type `Bool` and reject the definition. The closest we could come to the type signature of `foo` would be

```
bar :: (a -> a) -> (Char, Bool)
```

which is the same as

```
bar :: forall a. ((a -> a) -> (Char, Bool))
```

But this is very different from `foo`. The `forall` at the outermost level means that `bar` promises to work with any argument `f` as long as `f` has the shape `a -> a` for some type `a` unknown to `bar`. Contrast this with `foo`, where it's the argument `f` who promises to be of shape `a -> a` for all types `a` at the same time , and it's `foo` who makes use of that promise by choosing both `a = Char` and `a = Bool`.

Concerning nomenclature, simple polymorphic functions like `bar` are said to have a rank-1 type while the type `foo` is classified as **rank-2 type**. In general, a **rank-n type** is a function that has at least one rank-(n-1) argument but no arguments of even higher rank.

The theoretical basis for higher rank types is **System F**[6], also known as the second-order lambda calculus. We will detail it in the section System F[7] in order to better understand the meaning of `forall` and its placement like in `foo` and `bar`.

[4] https://en.wikipedia.org/wiki/Universal%20quantification

[5] The `UnicodeSyntax` extension allows you to use the symbol ∀instead of the `forall` keyword in your Haskell source code.

[6] https://en.wikipedia.org/wiki/System%20F

[7] Chapter 51.2 on page 372

Haskell98 is based on the Hindley-Milner[8] type system, which is a restriction of System F and does not support `forall` and rank-2 types or types of even higher rank. You have to enable the `RankNTypes`[9] language extension to make use of the full power of System F.

But of course, there is a good reason that Haskell98 does not support higher rank types: type inference for the full System F is undecidable, the programmer would have to write down all type signatures. Thus, the early versions of Haskell have adopted the Hindley-Milner type system which only offers simple polymorphic function but enables complete type inference in return. Recent advances in research have reduced the burden of writing type signatures and made rank-n types practical in current Haskell compilers.

51.1.3 runST

For the practical Haskell programmer, the ST monad[10] is probably the first example of a rank-2 type in the wild. Similar to the IO monad, it offers mutable references

```
newSTRef   :: a -> ST s (STRef s a)
readSTRef  :: STRef s a -> ST s a
writeSTRef :: STRef s a -> a -> ST s ()
```

and mutable arrays. The type variable `s` represents the state that is being manipulated. But unlike IO, these stateful computations can be used in pure code. In particular, the function

```
runST :: (forall s. ST s a) -> a
```

sets up the initial state, runs the computation, discards the state and returns the result. As you can see, it has a rank-2 type. Why?

The point is that mutable references should be local to one `runST`. For instance,

```
v   = runST (newSTRef "abc")
foo = runST (readSTRef v)
```

is wrong because a mutable reference created in the context of one `runST` is used again in a second `runST`. In other words, the result type `a` in `(forall s. ST s a) -> a` may not be a reference like `STRef s String` in the case of `v`. But the rank-2 type guarantees exactly that! Because the argument must be polymorphic in `s`, it has to return one and the same type `a` for all states `s`; the result `a` may not depend on the state. Thus, the unwanted code snippet above contains a type error and the compiler will reject it.

You can find a more detailed explanation of the ST monad in the original paper Lazy functional state threads[11][12].

[8] https://en.wikipedia.org/wiki/Hindley-Milner

[9] Or enable just `Rank2Types` if you only want to use rank-2 types

[10] http://www.haskell.org/haskellwiki/Monad/ST

[11] http://www.dcs.gla.ac.uk/fp/papers/lazy-functional-state-threads.ps.Z

[12] John Launchbury; Simon Peyton Jones 1994-??-??. Lazy functional state threads- ACM Press”. pp. 24-35 http://

51.1.4 Impredicativity

- *predicative* = type variables instantiated to *monotypes*. *impredicative* = also *polytypes*. Example: `length [id :: forall a . a -> a]` or `Just (id :: forall a. a -> a)`. Subtly different from higher-rank.
- relation of polymorphic types by their generality, i.e. 'isInstanceOf'.
- haskell-cafe: RankNTypes short explanation.[13]

51.2 System F

Section goal = a little bit lambda calculus foundation to prevent brain damage from implicit type parameter passing.

- System F = Basis for all this ∀-stuff.
- Explicit type applications i.e. `map Int (+1) [1,2,3]`. ∀ similar to the function arrow ->.
- Terms depend on types. Big Λ for type arguments, small λ for value arguments.

51.3 Examples

Section goal = enable reader to judge whether to use data structures with ∀in his <u>own</u> code.

- Church numerals, Encoding of arbitrary recursive types (positivity conditions): `&forall x. (F x -> x) -> x`
- Continuations, Pattern-matching: `maybe`, `either` and `foldr`

I.e. ∀ can be put to good use for implementing data types in Haskell.

51.4 Other forms of Polymorphism

Section goal = contrast polymorphism in OOP and stuff. how type classes fit in.

- *ad-hoc polymorphism* = different behavior depending on type s. => Haskell type classes.
- *parametric polymorphism* = ignorant of the type actually used. => ∀
- *subtyping*

51.5 Free Theorems

Section goal = enable reader to come up with free theorems. no need to prove them, intuition is enough.

- free theorems for parametric polymorphism.

[13] `http://thread.gmane.org/gmane.comp.lang.haskell.cafe/40508/focus=40610`

51.6 See also

- Luca Cardelli. On Understanding Types, Data Abstraction, and Polymorphism[14].

[14] `http://lucacardelli.name/Papers/OnUnderstanding.A4.pdf`

52 Existentially quantified types

Existential types, or 'existentials' for short, are a way of 'squashing' a group of types into one, single type.

Existentials are part of GHC's *type system extensions*. They aren't part of Haskell98, and as such you'll have to either compile any code that contains them with an extra command-line parameter of `-XExistentialQuantification`, or put `{-# LANGUAGE ExistentialQuantification #-}` at the top of your sources that use existentials.

52.1 The `forall` keyword

The `forall` keyword is used to explicitly bring fresh type variables into scope. For example, consider something you've innocuously seen written a hundred times so far:

Example: A polymorphic function

```
map :: (a -> b) -> [a] -> [b]
```

But what are these `a` and `b`? Well, they're type variables, you answer. The compiler sees that they begin with a lowercase letter and as such allows any type to fill that role. Another way of putting this is that those variables are 'universally quantified'. If you've studied formal logic, you will have undoubtedly come across the quantifiers: 'for all' (or $\forall$) and 'exists' (or $\exists$). They 'quantify' whatever comes after them: for example, $\exists x$ means that whatever follows is true for at least one value of x. $\forall x$ means that what follows is true for every possible value of x you can imagine. For example, $\forall x, x^2 \geq 0$ and $\exists x, x^3 = 27$.

The `forall` keyword quantifies *types* in a similar way. We would rewrite `map`'s type as follows:

Example: Explicitly quantifying the type variables

```
map :: forall a b. (a -> b) -> [a] -> [b]
```

So we see that for any combination of types `a` and `b` we can imagine, `map` takes the type `(a -> b) -> [a] -> [b]`. For example, we might choose `a = Int` and `b = String`. Then it's valid to say that `map` has the type `(Int -> String) -> [Int] -> [String]`. Here we are *instantiating* the general type of `map` to a more specific type.

However, in Haskell, any introduction of a lowercase type parameter implicitly begins with a `forall` keyword, so those two previous type declarations for `map` are equivalent, as are the declarations below:

Example: Two equivalent type statements

```
id :: a -> a
id :: forall a . a -> a
```

What makes life really interesting and the `forall` so useful is that you can apply additional constraints on the type variables it introduces. Such constraints, $P(x)$, serve to guarantee certain properties of the type variable, x, as a kind of ad-hoc interface restriction, (similar to $\exists x, P(x)$ or $\forall x, P(x)$ stipulations).

Let's dive right into the deep end of this by seeing an example of the power of existential types in action.

52.2 Example: heterogeneous lists

The premise behind Haskell's type class system is grouping types that all share a common property. So if you know a type that is a member of some class `C`, you know certain things about that type. For example, `Int` is a member of class `Eq`, so we know that elements of `Int` can be compared for equality.

Suppose we have a group of values. We don't know if they are all the same type, but we do know they are all members of some class (and, by extension, that all the values have a certain property). It might be useful to throw all these values into a list. We can't do this normally because lists elements must be of the same type (homogeneous with respect to types). However, existential types allow us to loosen this requirement by defining a 'type hider' or 'type box':

Example: Constructing a heterogeneous list

```
data ShowBox = forall s. Show s => SB s

heteroList :: [ShowBox]
heteroList = [SB (), SB 5, SB True]
```

We won't explain precisely what we mean by that data type definition, but its meaning should be clear to your intuition. The important thing is that we're calling the constructor on three values of different types, `[SB (), SB 5, SB True]`, yet we are able to place them all into a singe list, so we must somehow have the same type for each one. Essentially, yes. This is because our use of the `forall` keyword gives our constructor the type `SB :: forall s. Show s => s -> ShowBox`. If we were now writing a function to which we intend to pass `heteroList`, we couldn't apply a function such as `not` to the values inside the `SB` because their type might not be `Bool`. But we do know something about each of the elements: they can be converted to a string via `show`. In fact, that's pretty much the only thing we know about them.

Example: Using our heterogeneous list

```
instance Show ShowBox where
 show (SB s) = show s        -- (*) see the comment in the text below

f :: [ShowBox] -> IO ()
f xs = mapM_ print xs

main = f heteroList
```

Let's expand on this a bit more. In the definition of `show` for `ShowBox` – the line marked with `(*) see the comment in the text below` – we don't know the type of `s`. But as we mentioned, we *do* know that the type is an instance of Show due to the constraint on the `SB` constructor. Therefore, it's legal to use the function `show` on `s`, as seen in the right-hand side of the function definition.

As for `f`, recall the type of print:

Example: Types of the functions involved

```
print :: Show s => s -> IO () -- print x = putStrLn (show x)
mapM_ :: (a -> m b) -> [a] -> m ()
mapM_ print :: Show s => [s] -> IO ()
```

As we just declared `ShowBox` an instance of `Show`, we can print the values in the list.

52.3 A Further Explanation

One way to think about `forall` is to think about types as a set of possible values. For example, Bool is the set {True, False, ⊥} (remember that bottom, ⊥, is a member of every type!), Integer is the set of integers (and bottom), String is the set of all possible strings (and bottom), and so on. `forall` serves as a way to assert a commonality or *intersection* of the specified types (i.e. sets of values). For example, `forall a. a` is the intersection of all types. This subset turns out to be the set whose sole element is bottom, {⊥}, since it is an implicit value in every type. That is, the type whose only available value is bottom. However, since every Haskell type includes bottom, {⊥}, this quantification in fact stipulates all Haskell types. However, the only permissible operations on it are those available to a type whose only element is bottom.

A few more examples:

1. The list, `[forall a. a]`, is the type of a list whose elements all have the type `forall a. a`, i.e. a list of bottoms.
2. The list, `[forall a. Show a => a]`, is the type of a list whose elements all have the type `forall a. Show a => a`. The Show class constraint requires the possible types

to also be a member of the class, Show. However, ⊥ is still the only value common to all these types, so this too is a list of bottoms.

3. The list, `[forall a. Num a => a]`, requires each element to be a member of the class, Num. Consequently, the possible values include numeric literals, which have the specific type, `forall a. Num a => a`, as well as bottom.
4. `forall a. [a]` is the type of the list whose elements all have the same type `a`. Since we cannot presume any particular type at all, this too is a list of bottoms.

We see that most intersections over types just lead to bottoms because types generally don't have any values in common and so presumptions cannot be made about a union of their values.

However, recall that in the last section, we developed a heterogeneous list using a 'type hider'. This 'type hider' functions as a wrapper type which guarantees certain facilities by implying a predicate or constraint on the permissible types. In that case it was that they must be a member of the type class, Show. In general, that seems to be the purpose of `forall`, to impose type constraint on the permissible types within a type declaration and thereby guaranteeing certain facilities with such types.

Let's declare one.

Example: An existential datatype

```
data T = forall a. MkT a
```

This means that:

Example: This defines a family of constructors for T

```
  MkT :: forall a. (a -> T)
```

So we can pass any type, `a`, we want to `MkT` and it will create a T. So what happens when we deconstruct a `T` value with pattern matching...?

Example: Pattern matching on our existential constructor

```
foo (MkT x) = ... -- what is the type of x?
```

As we've just stated, `x` could be of any type. That means it's a member of some arbitrary type, so has the type `forall a. a`. In other words the set whose only available value is bottom, &prep;.

However, we can make a heterogeneous list:

Example: Constructing the hetereogeneous list

```
heteroList = [MkT 5, MkT (), MkT True, MkT map]
```

Of course, when we pattern match on `heteroList` we cannot presume any features about its elements[1]. So technically, we can't do anything *useful* with its elements, except reduce them to WHNF.because all we know is that they have some arbitrary type. However, if we introduce class constraints:

Example: A new existential data type, with a class constraint

```
data T' = forall a. Show a => MkT' a
```

The class constraint serves to limit the types we are intersecting over, such that we now have values inside a `T'` which are elements of some arbitrary type *that are members of Show*. The implication of this is that we can apply `show` to a value of type `a` upon deconstruction. It doesn't matter exactly which type it turns out to be.

Example: Using our new heterogenous setup

```
heteroList' = [MkT' 5, MkT' (), MkT' True, MkT' "Sartre"]
main = mapM_ (\(MkT' x) -> print x) heteroList'

{- prints:
5
()
True
"Sartre"
-}
```

To summaries, the interaction of the universal quantifier with data types produces a qualified subset of types guaranteeing certain facilities as described by one or more class constraints.

52.4 Example: `runST`

One monad that you may not have come across so far is the ST monad. This is essentially a more powerful version of the `State` monad: it has a much more complicated structure and involves some more advanced topics. It was originally written to provide Haskell with IO. As we mentioned in the ../Understanding monads/[2] chapter, IO is basically just a State monad with an environment of all the information about the real world. In fact, inside GHC at least, ST is used, and the environment is a type called `RealWorld`.

[1] However, we can apply them to functions whose type is `forall a. a -> R`, for some arbitrary *R*, as these accept values of any type as a parameter. Examples of such functions: `id`, `const k` for any `k`, `seq`

[2] Chapter 30 on page 179

To get out of the State monad, you can use `runState`. The analogous function for ST is called `runST`, and it has a rather particular type:

Example: The `runST` function

```
runST :: forall a. (forall s. ST s a) -> a
```

This is actually an example of a more complicated language feature called rank-2 polymorphism, which we don't go into in detail here. It's important to notice that there is no parameter for the initial state. Indeed, ST uses a different notion of state to State; while State allows you to `get` and `put` the current state, ST provides an interface to *references*. You create references, which have type `STRef`, with `newSTRef :: a -> ST s (STRef s a)`, providing an initial value, then you can use `readSTRef :: STRef s a -> ST s a` and `writeSTRef :: STRef s a -> a -> ST s ()` to manipulate them. As such, the internal environment of a ST computation is not one specific value, but a mapping from references to values. Therefore, you don't need to provide an initial state to runST, as the initial state is just the empty mapping containing no references.

However, things aren't quite as simple as this. What stops you creating a reference in one ST computation, then using it in another? We don't want to allow this because (for reasons of thread-safety) no ST computation should be allowed to assume that the initial internal environment contains any specific references. More concretely, we want the following code to be invalid:

Example: Bad ST code

```
let v = runST (newSTRef True)
in runST (readSTRef v)
```

What would prevent this? The effect of the rank-2 polymorphism in `runST`'s type is to *constrain the scope of the type variable* `s` to be within the first parameter. In other words, if the type variable `s` appears in the first parameter it cannot also appear in the second. Let's take a look at how exactly this is done. Say we have some code like the following:

Example: Briefer bad ST code

```
... runST (newSTRef True) ...
```

The compiler tries to fit the types together:

Example: The compiler's typechecking stage

```
newSTRef True :: forall s. ST s (STRef s Bool)
runST :: forall a. (forall s. ST s a) -> a
together, forall a. (forall s. ST s (STRef s Bool)) -> STRef s Bool
```

The importance of the `forall` in the first bracket is that we can change the name of the `s`. That is, we could write:

Example: A type mismatch!

```
together, forall a. (forall s'. ST s' (STRef s' Bool)) -> STRef s Bool
```

This makes sense: in mathematics, saying $\forall x.x > 5$ is precisely the same as saying $\forall y.y > 5$; you're just giving the variable a different label. However, we have a problem with our above code. Notice that as the `forall` does *not* scope over the return type of `runST`, we don't rename the `s` there as well. But suddenly, we've got a type mismatch! The result type of the ST computation in the first parameter must match the result type of `runST`, but now it doesn't!

The key feature of the existential is that it allows the compiler to generalise the type of the state in the first parameter, and so the result type cannot depend on it. This neatly sidesteps our dependence problems, and 'compartmentalises' each call to `runST` into its own little heap, with references not being able to be shared between different calls.

52.5 Quantification as a primitive

Universal quantification is useful for defining data types that aren't already defined. Suppose there was no such thing as pairs built into haskell. Quantification could be used to define them.

```
{-# LANGUAGE ExistentialQuantification, RankNTypes #-}

newtype Pair a b = Pair (forall c. (a -> b -> c) -> c)

makePair :: a -> b -> Pair a b
makePair a b = Pair $ \f -> f a b
```

In GHCi:

```
λ> :bro
newtype Pair a b = Pair {runPair :: forall c. (a -> b -> c) -> c}
makePair :: a -> b -> Pair a b
```

```
λ> let pair = makePair "a" 'b'

λ> :t pair
pair :: Pair [Char] Char

λ> runPair pair (\x y -> x)
"a"

λ> runPair pair (\x y -> y)
'b'
```

52.6 Further reading

- 24 Days of GHC Extensions: Existential Quantification[3]
- GHC's user guide contains useful information[4] on existentials, including the various limitations placed on them (which you should know about).
- *Lazy Functional State Threads[5], by Simon Peyton-Jones and John Launchbury, is a paper which explains more fully the ideas behind ST.*

[3] https://ocharles.org.uk/blog/guest-posts/2014-12-19-existential-quantification.html

[4] http://haskell.org/ghc/docs/latest/html/users_guide/data-type-extensions.html#existential-quantificatio

[5] http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.50.3299

53 Advanced type classes

Type classes may seem innocuous, but research on the subject has resulted in several advancements and generalisations which make them a very powerful tool.

53.1 Multi-parameter type classes

Multi-parameter type classes are a generalisation of the single parameter type classes[1], and are supported by some Haskell implementations.

Suppose we wanted to create a ’Collection’ type class that could be used with a variety of concrete data types, and supports two operations -- ’insert’ for adding elements, and ’member’ for testing membership. A first attempt might look like this:

Example: The `Collection` type class (wrong)

```
class Collection c where
    insert :: c -> e -> c
    member :: c -> e -> Bool

-- Make lists an instance of Collection:
instance Collection [a] where
    insert xs x = x:xs
    member = flip elem
```

This won’t compile, however. The problem is that the ’e’ type variable in the Collection operations comes from nowhere -- there is nothing in the type of an instance of Collection that will tell us what the ’e’ actually is, so we can never define implementations of these methods. Multi-parameter type classes solve this by allowing us to put ’e’ into the type of the class. Here is an example that compiles and can be used:

[1] `https://en.wikibooks.org/wiki/Classes%20and%20types`

Example: The `Collection` type class (right)

```
{-# LANGUAGE FlexibleInstances #-}
{-# LANGUAGE MultiParamTypeClasses #-}
class Eq e => Collection c e where
    insert :: c -> e -> c
    member :: c -> e -> Bool

instance Eq a => Collection [a] a where
    insert = flip (:)
    member = flip elem
```

53.2 Functional dependencies

A problem with the above example is that, in this case, we have extra information that the compiler doesn't know, which can lead to false ambiguities and over-generalised function signatures. In this case, we can see intuitively that the type of the collection will always determine the type of the element it contains - so if `c` is `[a]`, then `e` will be `a`. If `c` is `Hashmap a`, then `e` will be `a`. (The reverse is not true: many different collection types can hold the same element type, so knowing the element type was e.g. `Int`, would not tell you the collection type).

In order to tell the compiler this information, we add a **functional dependency**, changing the class declaration to

Example: A functional dependency

```
class Eq e => Collection c e | c -> e where ...
```

A functional dependency is a constraint that we can place on type class parameters. Here, the extra `| c -> e` should be read 'c uniquely identifies `e`', meaning for a given `c`, there will only be one `e`. You can have more than one functional dependency in a class -- for example you could have `c -> e, e -> c` in the above case. And you can have more than two parameters in multi-parameter classes.

53.2.1 Examples

Matrices and vectors

Suppose you want to implement some code to perform simple linear algebra:

Example: The `Vector` and `Matrix` datatypes

```
data Vector = Vector Int Int deriving (Eq, Show)
data Matrix = Matrix Vector Vector deriving (Eq, Show)
```

You want these to behave as much like numbers as possible. So you might start by overloading Haskell's Num class:

Example: Instance declarations for `Vector` and `Matrix`

```
instance Num Vector where
  Vector a1 b1 + Vector a2 b2 = Vector (a1+a2) (b1+b2)
  Vector a1 b1 - Vector a2 b2 = Vector (a1-a2) (b1-b2)
  {- ... and so on ... -}

instance Num Matrix where
  Matrix a1 b1 + Matrix a2 b2 = Matrix (a1+a2) (b1+b2)
  Matrix a1 b1 - Matrix a2 b2 = Matrix (a1-a2) (b1-b2)
  {- ... and so on ... -}
```

The problem comes when you want to start multiplying quantities. You really need a multiplication function which overloads to different types:

Example: What we need

```
(*) :: Matrix -> Matrix -> Matrix
(*) :: Matrix -> Vector -> Vector
(*) :: Matrix -> Int -> Matrix
(*) :: Int -> Matrix -> Matrix
{- ... and so on ... -}
```

How do we specify a type class which allows all these possibilities?

We could try this:

Example: An ineffective attempt (too general)

```
class Mult a b c where
  (*) :: a -> b -> c

instance Mult Matrix Matrix Matrix where
  {- ... -}

instance Mult Matrix Vector Vector where
  {- ... -}
```

That, however, isn't really what we want. As it stands, even a simple expression like this has an ambiguous type unless you supply an additional type declaration on the intermediate expression:

Example: Ambiguities lead to more verbose code

```
m1, m2, m3 :: Matrix
(m1 * m2) * m3              -- type error; type of (m1*m2) is ambiguous
(m1 * m2) :: Matrix * m3    -- this is ok
```

After all, nothing is stopping someone from coming along later and adding another instance:

Example: A nonsensical instance of `Mult`

```
instance Mult Matrix Matrix (Maybe Char) where
  {- whatever -}
```

The problem is that `c` shouldn't really be a free type variable. When you know the types of the things that you're multiplying, the result type should be determined from that information alone.

You can express this by specifying a functional dependency:

Example: The correct definition of `Mult`

```
class Mult a b c | a b -> c where
  (*) :: a -> b -> c
```

This tells Haskell that `c` is uniquely determined from `a` and `b`.

54 Phantom types

Phantom types are a way to embed a language with a stronger type system than Haskell's.

54.1 Phantom types

An ordinary type

```
data T = TI Int | TS String

plus :: T -> T -> T
concat :: T -> T -> T
```

its phantom type version

```
data T a = TI Int | TS String
```

Nothing's changed - just a new argument `a` that we don't touch. But magic!

```
plus :: T Int -> T Int -> T Int
concat :: T String -> T String -> T String
```

Now we can enforce a little bit more!

This is useful if you want to increase the type-safety of your code, but not impose additional runtime overhead:

```
-- Peano numbers at the type level.
data Zero = Zero
data Succ a = Succ a
-- Example: 3 can be modeled as the type
-- Succ (Succ (Succ Zero)))

type D2 = Succ (Succ Zero)
type D3 = Succ (Succ (Succ Zero))

data Vector n a = Vector [a] deriving (Eq, Show)

vector2d :: Vector D2 Int
vector2d = Vector [1,2]

vector3d :: Vector D3 Int
vector3d = Vector [1,2,3]
```

```
-- vector2d == vector3d raises a type error
-- at compile-time:

--   Couldn't match expected type `Zero'
--               with actual type `Succ Zero'
--   Expected type: Vector D2 Int
--     Actual type: Vector D3 Int
--   In the second argument of `(==)', namely `vector3d'
--   In the expression: vector2d == vector3d

-- while vector2d == Vector [1,2,3] works
```

55 Generalised algebraic data-types (GADT)

w:Generalized algebraic data type[1]

55.1 Introduction

Generalized algebraic datatypes, or simply GADTs, are a generalization of the algebraic data types that you are familiar with. Basically, they allow you to explicitly write down the types of the constructors. In this chapter, you'll learn why this is useful and how to declare your own.

We begin with an example of building a simple embedded domain specific language (EDSL) for simple arithmetical expressions, which is put on a sounder footing with GADTs. This is followed by a review of the syntax for GADTs, with simpler illustrations, and a different application to construct a safe list type for which the equivalent of `head []` fails to typecheck and thus does not give the usual runtime error: `*** Exception: Prelude.head: empty list`.

55.2 Understanding GADTs

So, what are GADTs and what are they useful for? GADTs are mainly used to implement domain specific languages, and so this section will introduce them with a corresponding example.

55.2.1 Arithmetic expressions

Let's consider a small language for arithmetic expressions, given by the data type

```
data Expr = I Int         -- integer constants
          | Add Expr Expr -- add two expressions
          | Mul Expr Expr -- multiply two expressions
```

In other words, this data type corresponds to the abstract syntax tree, an arithmetic term like `(5+1)*7` would be represented as ``(I 5 `Add` I 1) `Mul` I 7 :: Expr``.

[1] https://en.wikipedia.org/wiki/Generalized%20algebraic%20data%20type

Given the abstract syntax tree, we would like to do something with it; we want to compile it, optimize it and so on. For starters, let's write an evaluation function that takes an expression and calculates the integer value it represents. The definition is straightforward:

```
eval :: Expr -> Int
eval (I n)       = n
eval (Add e1 e2) = eval e1 + eval e2
eval (Mul e1 e2) = eval e1 * eval e2
```

55.2.2 Extending the language

Now, imagine that we would like to extend our language with other types than just integers. For instance, let's say we want to represent equality tests, so we need booleans as well. We augment the ‘Expr‘ type to become

```
data Expr = I Int
          | B Bool           -- boolean constants
          | Add Expr Expr
          | Mul Expr Expr
          | Eq  Expr Expr    -- equality test
```

The term `5+1 == 7` would be represented as ``(I 5 `Add` I 1) `Eq` I 7``.

As before, we want to write a function `eval` to evaluate expressions. But this time, expressions can now represent either integers or booleans and we have to capture that in the return type

```
eval :: Expr -> Either Int Bool
```

The first two cases are straightforward

```
eval (I n) = Left n
eval (B b) = Right b
```

but now we get in trouble. We would like to write

```
eval (Add e1 e2) = eval e1 + eval e2  -- ???
```

but this doesn't type check: the addition function `+` expects two integer arguments, but `eval e1` is of type `Either Int Bool` and we'd have to extract the `Int` from that.

Even worse, what happens if `e1` actually represents a *boolean*? The following is a valid expression

```
B True `Add` I 5 :: Expr
```

but clearly, it doesn't make any sense; we can't add booleans to integers! In other words, evaluation may return integers or booleans, but it may also *fail* because the expression makes no sense. We have to incorporate that in the return type:

```
eval :: Expr -> Maybe (Either Int Bool)
```

Now, we could write this function just fine, but that would still be unsatisfactory, because what we *really* want to do is to have Haskell's type system rule out any invalid expressions; we don't want to check types ourselves while deconstructing the abstract syntax tree.

Exercise: Despite our goal, it may still be instructional to implement the `eval` function; do this.

Starting point:

```
data Expr = I Int
        | B Bool           -- boolean constants
        | Add Expr Expr
        | Mul Expr Expr
        | Eq  Expr Expr    -- equality test

eval :: Expr -> Maybe (Either Int Bool)
-- Your implementation here.
```

55.2.3 Phantom types

The so-called *phantom types* are the first step towards our goal. The technique is to augment the `Expr` with a type variable, so that it becomes

```
data Expr a = I Int
            | B Bool
            | Add (Expr a) (Expr a)
            | Mul (Expr a) (Expr a)
            | Eq  (Expr a) (Expr a)
```

Note that an expression `Expr a` does not contain a value `a` at all; that's why `a` is called a *phantom type*, it's just a dummy variable. Compare that with, say, a list `[a]` which does contain a bunch of `a`'s.

The key idea is that we're going to use `a` to track the type of the expression for us. Instead of making the constructor

```
Add :: Expr a -> Expr a -> Expr a
```

available to users of our small language, we are only going to provide a *smart constructor* with a more restricted type

```
add :: Expr Int -> Expr Int -> Expr Int
add = Add
```

The implementation is the same, but the types are different. Doing this with the other constructors as well,

```
i :: Int  -> Expr Int
i = I
b :: Bool -> Expr Bool
b = B
```

the previously problematic expression

```
b True `add` i 5
```

no longer type checks! After all, the first arguments has the type `Expr Bool` while `add` expects an `Expr Int`. In other words, the phantom type `a` marks the intended type of the expression. By only exporting the smart constructors, the user cannot create expressions with incorrect types.

As before, we want to implement an evaluation function. With our new marker `a`, we might hope to give it the type

```
eval :: Expr a -> a
```

and implement the first case like this

```
eval (I n) = n
```

But alas, this does not work: how would the compiler know that encountering the constructor `I` means that `a = Int`? Granted, this will be case for all the expression that were created by users of our language because they are only allowed to use the smart constructors. But internally, an expression like

```
I 5 :: Expr String
```

is still valid. In fact, as you can see, `a` doesn't even have to be `Int` or `Bool`, it could be anything.

What we need is a way to restrict the return types of the constructors themselves, and that's exactly what generalized data types do.

55.2.4 GADTs

The obvious notation for restricting the type of a constructor is to write down its type, and that's exactly how GADTs are defined:

```
data Expr a where
    I   :: Int  -> Expr Int
    B   :: Bool -> Expr Bool
    Add :: Expr Int -> Expr Int -> Expr Int
    Mul :: Expr Int -> Expr Int -> Expr Int
    Eq  :: Expr Int -> Expr Int -> Expr Bool
```

In other words, we simply list the type signatures of all the constructors. In particular, the marker type `a` is specialised to `Int` or `Bool` according to our needs, just like we would have done with smart constructors.

And the great thing about GADTs is that we now *can* implement an evaluation function that takes advantage of the type marker:

```
eval :: Expr a -> a
eval (I n) = n
eval (B b) = b
eval (Add e1 e2) = eval e1 + eval e2
eval (Mul e1 e2) = eval e1 * eval e2
eval (Eq  e1 e2) = eval e1 == eval e2
```

In particular, in the first case

```
eval (I n) = n
```

the compiler is now able infer that `a=Int` when we encounter a constructor `I` and that it is legal to return the `n :: Int`; similarly for the other cases.

To summarise, GADTs allows us to restrict the return types of constructors and thus enable us to take advantage of Haskell's type system for our domain specific languages. Thus, we can implement more languages and their implementation becomes simpler.

55.3 Summary

55.3.1 Syntax

Here a quick summary of how the syntax for declaring GADTs works.

First, consider the following ordinary algebraic datatypes: the familiar `List` and `Maybe` types, and a simple tree type, `RoseTree`:

Maybe

```
data Maybe a =
    Nothing |
    Just a
```

List

```
data List a =
    Nil |
    Cons a (List a)
```

Rose Tree

```
data RoseTree a =
     RoseTree a [RoseTree a]
```

Remember that the constructors introduced by these declarations can be used both for pattern matches to deconstruct values and as functions to construct values. (`Nothing` and `Nil` are functions with "zero arguments".) We can ask what the types of the latter are:

Maybe

```
> :t Nothing
Nothing :: Maybe a
> :t Just
Just :: a -> Maybe a
```

List

```
> :t Nil
Nil :: List a
> :t Cons
Cons :: a -> List a -> List a
```

Rose Tree

```
> :t RoseTree
RoseTree ::
   a -> [RoseTree a] -
> RoseTree a
```

It is clear that this type information about the constructors for `Maybe`, `List` and `RoseTree` respectively is equivalent to the information we gave to the compiler when declaring the datatype in the first place. In other words, it's also conceivable to declare a datatype by simply listing the types of all of its constructors, and that's exactly what the GADT syntax does:

Maybe

```
data Maybe a where
   Nothing  :: Maybe a
   Just :: a -> Maybe a
```

List

```
data List a where
   Nil  :: List a
   Cons :: a -> List a -
> List a
```

Rose Tree

```
data RoseTree a where
   RoseTree ::
      a ->  [RoseTree a] -
> RoseTree a
```

This syntax is made available by the language option {-#LANGUAGE GADTs #-}. It should be familiar to you in that it closely resembles the syntax of type class declarations. It's also easy to remember if you already like to think of constructors as just being functions. Each constructor is just defined by a type signature.

55.3.2 New possibilities

Note that when we asked the GHCi for the types of Nothing and Just it returned Maybe a and a -> Maybe a as the types. In these and the other cases, the type of the final output of the function associated with a constructor is the type we were initially defining - Maybe a, List a or RoseTree a. In general, in standard Haskell, the constructor functions for Foo a have Foo a as their final return type. If the new syntax were to be strictly equivalent to the old, we would have to place this restriction on its use for valid type declarations.

So what do GADTs add for us? The ability to control exactly what kind of Foo you return. With GADTs, a constructor for Foo a is not obliged to return Foo a; it can return any Foo blah that you can think of. In the code sample below, for instance, the GadtedFoo constructor returns a GadtedFoo Int even though it is for the type GadtedFoo x.

Example: GADT gives you more control

```
data FooInGadtClothing a where
 MkFooInGadtClothing :: a -> FooInGadtClothing a

--which is no different from:  data Haskell98Foo a = MkHaskell98Foo a ,

--by contrast, consider:

data TrueGadtFoo a where
  MkTrueGadtFoo :: a -> TrueGadtFoo Int

--This has no Haskell 98 equivalent.
```

But note that you can only push the generalization so far... if the datatype you are declaring is a Foo, the constructor functions *must* return some kind of Foo or another. Returning anything else simply wouldn't work

Example: Try this out. It doesn't work

```
data Bar where
  BarNone :: Bar -- This is ok

data Foo where
  MkFoo :: Bar Int-- This will not typecheck
```

55.4 Examples

55.4.1 Safe Lists

Prerequisite: *We assume in this section that you know how a List tends to be represented in functional languages*

Note: *The examples in this section additionally require the extensions EmptyDataDecls and KindSignatures to be enabled*

We've now gotten a glimpse of the extra control given to us by the GADT syntax. The only thing new is that you can control exactly what kind of data structure you return. Now, what can we use it for? Consider the humble Haskell list. What happens when you invoke `head []`? Haskell blows up. Have you ever wished you could have a magical version of `head` that only accepts lists with at least one element, lists on which it will never blow up?

To begin with, let's define a new type, `SafeList x y`. The idea is to have something similar to normal Haskell lists `[x]`, but with a little extra information in the type. This extra information (the type variable `y`) tells us whether or not the list is empty. Empty lists are represented as `SafeList x Empty`, whereas non-empty lists are represented as `SafeList x NonEmpty`.

```
-- we have to define these types
data Empty
data NonEmpty

-- the idea is that you can have either
--    SafeList a Empty
-- or SafeList a NonEmpty
data SafeList a b where
-- to be implemented
```

Since we have this extra information, we can now define a function `safeHead` on only the non-empty lists! Calling `safeHead` on an empty list would simply refuse to type-check.

```
safeHead :: SafeList a NonEmpty -> a
```

So now that we know what we want, `safeHead`, how do we actually go about getting it? The answer is GADT. The key is that we take advantage of the GADT feature to return two *different* list-of-a types, `SafeList a Empty` for the `Nil` constructor, and `SafeList a NonEmpty` for the `Cons` constructor:

```
data SafeList a b where
  Nil  :: SafeList a Empty
  Cons :: a -> SafeList a b -> SafeList a NonEmpty
```

This wouldn't have been possible without GADT, because all of our constructors would have been required to return the same type of list; whereas with GADT we can now return different types of lists with different constructors. Anyway, let's put this all together, along with the actual definition of `SafeHead`:

Example: safe lists via GADT

```
{-LANGUAGE GADTs, EmptyDataDecls -}
-- (the EmptyDataDecls pragma must also appear at the very top of the module,
-- in order to allow the Empty and NonEmpty datatype declarations.)

data Empty
data NonEmpty

data SafeList a b where
     Nil :: SafeList a Empty
     Cons:: a -> SafeList a b -> SafeList a NonEmpty

safeHead :: SafeList a NonEmpty -> a
safeHead (Cons x _) = x
```

Copy this listing into a file and load in `ghci -fglasgow-exts`. You should notice the following difference, calling `safeHead` on a non-empty and an empty-list respectively:

Example: `safeHead` is... safe

```
Prelude Main> safeHead (Cons "hi" Nil)
"hi"
Prelude Main> safeHead Nil

<interactive>:1:9:
    Couldn't match `NonEmpty' against `Empty'
      Expected type: SafeList a NonEmpty
      Inferred type: SafeList a Empty
    In the first argument of `safeHead', namely `Nil'
    In the definition of `it': it = safeHead Nil
```

The complaint is a good thing: it means that we can now ensure during compile-time if we're calling `safeHead` on an appropriate list. However, that also sets up a pitfall in potential. Consider the following function. What do you think its type is?

Example: Trouble with GADTs

```
silly False = Nil
silly True  = Cons () Nil
```

Now try loading the example up in GHCi. You'll notice the following complaint:

Example: Trouble with GADTs - the complaint

```
Couldn't match `Empty' against `NonEmpty'
     Expected type: SafeList () Empty
     Inferred type: SafeList () NonEmpty
   In the application `Cons () Nil'
   In the definition of `silly': silly True = Cons () Nil
```

The cases in the definition of `silly` evaluate to marked lists of different types, leading to a type error. The extra constraints imposed through the GADT make it impossible for a function to produce both empty and non-empty lists.

If we are really keen on defining `silly`, we can do so by liberalizing the type of `Cons`, so that it can construct both safe and unsafe lists.

Example: A different approach

```
{-LANGUAGE GADTs, EmptyDataDecls, KindSignatures -}
-- here we add the KindSignatures pragma,
-- which makes the GADT declaration a bit more elegant.

-- Note the subtle yet revealing change in the phantom type names.
data NotSafe
data Safe



data MarkedList             ::  * -> * -> * where
  Nil                       ::  MarkedList t NotSafe
  Cons                      ::  a -> MarkedList a b -> MarkedList a c



safeHead                    ::  MarkedList a Safe -> a
safeHead (Cons x _)          =  x

-- This function will never produce anything that can be consumed by safeHead,
-- no matter that the resulting list is not necessarily empty.
silly                       :: Bool -> MarkedList () NotSafe
silly False                  =  Nil
silly True                   =  Cons () Nil
```

There is a cost to the fix above: by relaxing the constraint on `Cons` we throw away the knowledge that it cannot produce an empty list. Based on our first version of the safe list we could, for instance, define a function which took a `SafeList a Empty` argument and be sure anything produced by `Cons` would not be accepted by it. That does not hold for the analogous `MarkedList a NotSafe`; arguably, the type is less useful exactly because it is less restrictive. While in this example the issue may seem minor, given that not much can be done with an empty list, in general it is worth considering.

Exercises:

1. Could you implement a `safeTail` function? Both versions introduced here would count as valid starting material, as well as any other variants in similar spirit.

55.4.2 A simple expression evaluator

Insert the example used in Wobbly Types paper... I thought that was quite pedagogical

This is already covered in the first part of the tutorial.

55.5 Discussion

More examples, thoughts

From FOSDEM 2006, I vaguely recall that there is some relationship between GADT and the below... what?

55.5.1 Phantom types

See ../Phantom types/[2].

55.5.2 Existential types

If you like ../Existentially quantified types/[3], you'd probably want to notice that they are now subsumed by GADTs. As the GHC manual says, the following two type declarations give you the same thing.

```
data TE a = forall b. MkTE b (b->a)
data TG a where { MkTG :: b -> (b->a) -> TG a }
```

Heterogeneous lists are accomplished with GADTs like this:

```
data TE2 = forall b. Show b => MkTE2 [b]
data TG2 where
  MkTG2 :: Show b => [b] -> TG2
```

55.5.3 Witness types

[2] Chapter 54 on page 387

[3] Chapter 52 on page 375

56 Type constructors & Kinds

56.1 Kinds for C++ users

- * is any concrete type, including functions. These all have kind *:

```
type MyType = Int
type MyFuncType = Int -> Int
myFunc :: Int -> Int
```

```
typedef int MyType;
typedef int (*MyFuncType)(int);
int MyFunc(int a);
```

- * -> * is a template that takes one type argument. It is like a function from types to types: you plug a type in and the result is a type. Confusion can arise from the two uses of MyData (although you can give them different names if you wish) - the first is a type constructor, the second is a data constructor. These are equivalent to a class template and a constructor respectively in C++. Context resolves the ambiguity - where Haskell expects a type (e.g. in a type signature) MyData is a type constructor, where a value, it is a data constructor.

```
data MyData t -- type constructor with kind * -> *
              = MyData t -- data constructor with type a -> MyData a
*Main> :k MyData
MyData :: * -> *
*Main> :t MyData
MyData :: a -> MyData a
```

```
template <typename t> class MyData
{
   t member;
};
```

- * -> * -> * is a template that takes two type arguments

```
data MyData t1 t2 = MyData t1 t2
```

```
template <typename t1, typename t2> class MyData
{
   t1 member1;
   t2 member2;
   MyData(t1 m1, t2 m2) : member1(m1), member2(m2) { }
};
```

- (* -> *) -> * is a template that takes one template argument of kind (* -> *)

```
data MyData tmpl = MyData (tmpl Int)
```

```
template <template <typename t> class tmpl> class MyData
{
   tmpl<int> member1;
   MyData(tmpl<int> m) : member1(m) { }
};
```

57 Wider Theory

58 Denotational semantics

New readers: Please report stumbling blocks! While the material on this page is intended to explain clearly, there are always mental traps that innocent readers new to the subject fall in but that the authors are not aware of. Please report any tricky passages to the Talk[1] page or the #haskell IRC channel so that the style of exposition can be improved.

58.1 Introduction

This chapter explains how to formalize the meaning of Haskell programs, the **denotational semantics**. It may seem to be nit-picking to formally specify that the program `square x = x*x` means the same as the mathematical square function that maps each number to its square, but what about the meaning of a program like `f x = f (x+1)` that loops forever? In the following, we will exemplify the approach first taken by Scott and Strachey to this question and obtain a foundation to reason about the correctness of functional programs in general and recursive definitions in particular. Of course, we will concentrate on those topics needed to understand Haskell programs.[2]

Another aim of this chapter is to illustrate the notions **strict** and **lazy** that capture the idea that a function needs or needs not to evaluate its argument. This is a basic ingredient to predict the course of evaluation of Haskell programs and hence of primary interest to the programmer. Interestingly, these notions can be formulated concisely with denotational semantics alone, no reference to an execution model is necessary. They will be put to good use in Graph Reduction[3], but it is this chapter that will familiarize the reader with the denotational definition and involved notions such as ⊥ (”Bottom”). The reader only interested in strictness may wish to poke around in section Bottom and Partial Functions[4] and quickly head over to Strict and Non-Strict Semantics[5].

58.1.1 What are Denotational Semantics and what are they for?

What does a Haskell program mean? This question is answered by the **denotational semantics** of Haskell. In general, the denotational semantics of a programming language map each of its programs to a mathematical object (denotation), that represents

[2] In fact, there are no written down and complete denotational semantics of Haskell. This would be a tedious task void of additional insight and we happily embrace the folklore and common sense semantics.

[3] Chapter 65 on page 463

[4] Chapter 58.2 on page 408

[5] Chapter 58.4 on page 417

the *meaning* of the program in question. As an example, the mathematical object for the Haskell programs `10`, `9+1`, `2*5` and `sum [1..4]` can be represented by the integer *10*. We say that all those programs **denote** the integer *10*. The collection of such mathematical objects is called the **semantic domain**.

The mapping from program code to a semantic domain is commonly written down with double square brackets (”Oxford brackets”) around program code. For example,

$$[[\texttt{2*5}]] = 10.$$

Denotations are *compositional*, i.e. the meaning of a program like `1+9` only depends on the meaning of its constituents:

$$[[\texttt{a+b}]] = [[\texttt{a}]] + [[\texttt{b}]].$$

The same notation is used for types, i.e.

$$[[\texttt{Integer}]] = \mathbb{Z}.$$

For simplicity however, we will silently identify expressions with their semantic objects in subsequent chapters and use this notation only when clarification is needed.

It is one of the key properties of *purely functional* languages like Haskell that a direct mathematical interpretation like ”`1+9` denotes *10*” carries over to functions, too: in essence, the denotation of a program of type `Integer -> Integer` is a mathematical function $\mathbb{Z} \to \mathbb{Z}$ between integers. While we will see that this expression needs refinement generally, to include non-termination, the situation for *imperative languages* is clearly worse: a procedure with that type denotes something that changes the state of a machine in possibly unintended ways. Imperative languages are tightly tied to **operational semantics** which describes their way of execution on a machine. It is possible to define a denotational semantics for imperative programs and to use it to reason about such programs, but the semantics often has operational nature and sometimes must be extended in comparison to the denotational semantics for functional languages.[6] In contrast, the meaning of purely functional languages is *by default* completely independent from their way of execution. The Haskell98 standard even goes as far as to specify only Haskell’s non-strict denotational semantics, leaving open how to implement them.

In the end, denotational semantics enables us to develop formal proofs that programs indeed do what we want them to do mathematically. Ironically, for proving program properties in day-to-day Haskell, one can use Equational reasoning[7], which transforms programs into

[6] Monads are one of the most successful ways to give denotational semantics to imperative programs. See also Haskell/Advanced monads ^{https://en.wikibooks.org/wiki/Haskell%2FAdvanced%20monads} .

[7] https://en.wikibooks.org/wiki/Haskell%2FEquational%20reasoning

equivalent ones without seeing much of the underlying mathematical objects we are concentrating on in this chapter. But the denotational semantics actually show up whenever we have to reason about non-terminating programs, for instance in Infinite Lists[8].

Of course, because they only state what a program is, denotational semantics cannot answer questions about how long a program takes or how much memory it eats; this is governed by the *evaluation strategy* which dictates how the computer calculates the normal form of an expression. On the other hand, the implementation has to respect the semantics, and to a certain extent, it is the semantics that determine how Haskell programs must be evaluated on a machine. We will elaborate on this in Strict and Non-Strict Semantics[9].

58.1.2 What to choose as Semantic Domain?

We are now looking for suitable mathematical objects that we can attribute to every Haskell program. In case of the example 10, 2*5 and sum [1..4], it is clear that all expressions should denote the integer *10*. Generalizing, every value x of type Integer is likely to denote an element of the set $\mathbb{Z}$. The same can be done with values of type Bool. For functions like f :: Integer -> Integer, we can appeal to the mathematical definition of "function" as a set of (argument,value)-pairs, its *graph*.

But interpreting functions as their graph was too quick, because it does not work well with recursive definitions. Consider the definition

```
shaves :: Integer -> Integer -> Bool
1 `shaves` 1 = True
2 `shaves` 2 = False
0 `shaves` x = not (x `shaves` x)
_ `shaves` _ = False
```

We can think of 0,1 and 2 as being male persons with long beards and the question is who shaves whom. Person 1 shaves himself, but 2 gets shaved by the barber 0 because evaluating the third equation yields 0 `shaves` 2 == True. In general, the third line says that the barber 0 shaves all persons that do not shave themselves.

What about the barber himself, is 0 `shaves` 0 true or not? If it is, then the third equation says that it is not. If it is not, then the third equation says that it is. Puzzled, we see that we just cannot attribute True or False to 0 `shaves` 0, the graph we use as interpretation for the function shaves must have an empty spot. We realize that our semantic objects must be able to incorporate **partial functions**, functions that are undefined for some values of their arguments (..that is otherwise permitted by the arguments' types).

It is well known that this famous example gave rise to serious foundational problems in set theory. It's an example of an **impredicative** definition, a definition which uses itself, a logical circle. Unfortunately for recursive definitions, the circle is not the problem but the feature.

[8] Chapter 58.5.3 on page 424

[9] Chapter 58.4 on page 417

58.2 Bottom and Partial Functions

58.2.1 ⊥ Bottom

To define partial functions, we introduce a special value ⊥, named **bottom**[10] and commonly written `_|_` in typewriter font. We say that ⊥is the completely **"undefined" value or function**. Every basic data type like `Integer` or `()` contains one ⊥besides their usual elements. So the possible values of type `Integer` are

$$\perp, 0, +1, -1, +2, -2, +3, -3, \ldots$$

Adding ⊥to the set of values is also called **lifting**. This is often depicted by a subscript like in $\mathbb{Z}_\perp$. While this is the correct notation for the mathematical set "lifted integers", we prefer to talk about "values of type `Integer`". We do this because $\mathbb{Z}_\perp$ suggests that there are "real" integers $\mathbb{Z}$, but inside Haskell, the "integers" are `Integer`.

As another example, the type `()` with only one element actually has two inhabitants:

$$\perp, ()$$

For now, we will stick to programming with `Integers`. Arbitrary algebraic data types will be treated in section Algebraic Data Types[11] since strict and non-strict languages diverge on how these include ⊥.

In Haskell, the expression `undefined` denotes ⊥. With its help, one can indeed verify some semantic properties of actual Haskell programs. `undefined` has the polymorphic type `forall a . a` which of course can be specialized to `undefined :: Integer`, `undefined :: ()`, `undefined :: Integer -> Integer` and so on. In the Haskell Prelude, it is defined as

```
undefined = error "Prelude.undefined"
```

As a side note, it follows from the Curry-Howard isomorphism[12] that any value of the polymorphic type `forall a . a` must denote ⊥.

58.2.2 Partial Functions and the Semantic Approximation Order

Now, ⊥ (*bottom type*) gives us the possibility to denote partial functions:

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ -2 & \text{if } n \text{ is } 1 \\ \perp & \text{else} \end{cases}$$

[10] https://en.wikibooks.org/wiki/%3Aw%3ABottom%20type

[11] Chapter 58.5 on page 419

[12] https://en.wikibooks.org/wiki/Haskell%2FThe%20Curry-Howard%20isomorphism

Here, $f(n)$ yields well defined values for $n = 0$ and $n = 1$ but gives $\perp$ for all other n. Note that the type $\perp$ is universal, as $\perp$ has no value: the function $\perp$`:: Integer -> Integer` is given by

$$\perp(n) = \perp$$

for all

$$n$$

where the $\perp$ on the right hand side denotes a value of type `Integer`.

To formalize, **partial functions** say, of type `Integer -> Integer` are at least mathematical mappings from the lifted integers $\mathbb{Z}_\perp = \{\perp, 0, \pm 1, \pm 2, \pm 3, \dots\}$ to the lifted integers. But this is not enough, since it does not acknowledge the special role of $\perp$. For example, the definition

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is } \perp \\ \perp & \text{else} \end{cases}$$

looks counterintuitive, and, in fact, is wrong. Why does $g(\perp)$ yield a defined value whereas $g(1)$ is undefined? The intuition is that every partial function g should yield more defined answers for more defined arguments. To formalize, we can say that every concrete number is **more defined** than $\perp$:

$$\perp \sqsubset 1\ ,\ \perp \sqsubset 2\ , \dots$$

Here, $a \sqsubset b$ denotes that b is more defined than a. Likewise, $a \sqsubseteq b$ will denote that either b is more defined than a or both are equal (and so have the same definedness). $\sqsubset$ is also called the **semantic approximation order** because we can approximate defined values by less defined ones thus interpreting "more defined" as "approximating better". Of course, $\perp$ is designed to be the least element of a data type, we always have that $\perp \sqsubset x$ for all x, except the case when x happens to denote $\perp$ itself:

$$\forall x \neq \perp \quad \perp \sqsubset x$$

As no number is *more defined* than another, the mathematical relation $\sqsubset$ is false for any pair of numbers:

$$1 \sqsubset 1$$

does not hold.

neither

$$1 \sqsubset 2$$

nor

$$2 \sqsubset 1$$

hold.

This is contrasted to ordinary order predicate $\leq$, which can compare any two numbers. A quick way to remember this is the sentence: ”1 and 2 are different in terms of *information content* but are equal in terms of *information quantity*”. That’s another reason why we use a different symbol: $\sqsubseteq$.

neither

$$1 \sqsubseteq 2$$

nor

$$2 \sqsubseteq 1$$

hold,

but

$$1 \sqsubseteq 1$$

holds.

One says that $\sqsubseteq$ specifies a **partial order** and that the values of type `Integer` form a **partially ordered set** (**poset** for short). A partial order is characterized by the following three laws

- *Reflexivity*, everything is just as defined as itself: $x \sqsubseteq x$ for all x
- *Transitivity*: if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$
- *Antisymmetry*: if both $x \sqsubseteq y$ and $y \sqsubseteq x$ hold, then x and y must be equal: $x = y$.

Exercises:
Do the integers form a poset with respect to the order $\leq$?

We can depict the order $\sqsubseteq$ on the values of type `Integer` by the following graph

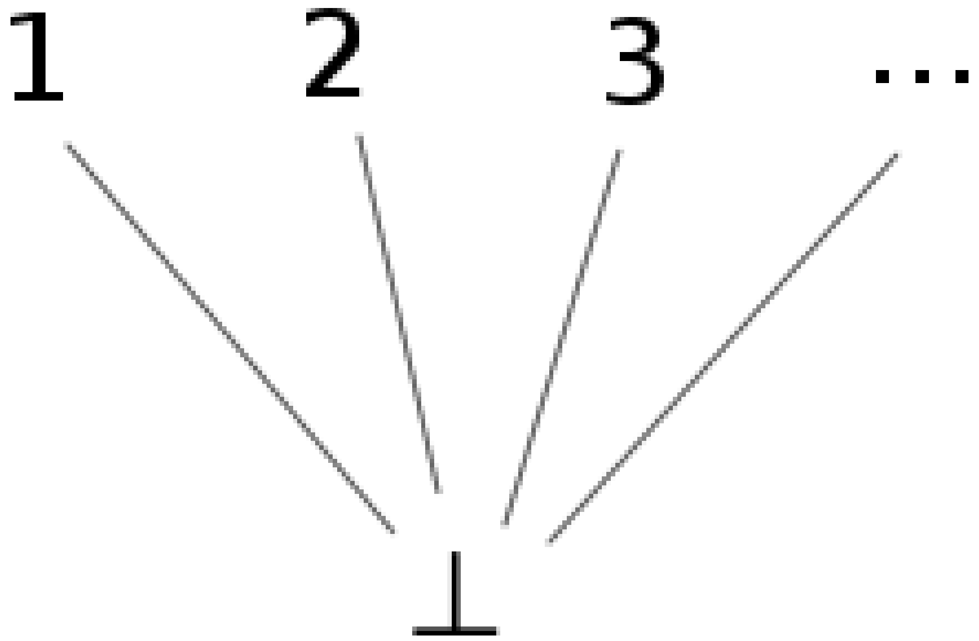


Figure 24

where every link between two nodes specifies that the one above is more defined than the one below. Because there is only one level (excluding $\perp$), one says that `Integer` is a *flat domain*. The picture also explains the name of $\perp$: it's called *bottom* because it always sits at the bottom.

58.2.3 Monotonicity

Our intuition about partial functions now can be formulated as following: every partial function f is a **monotone** mapping between partially ordered sets. More defined arguments will yield more defined values:

$$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$$

In particular, a function h with $h(\perp) = 1$ is constant: $h(n) = 1$ for all n. Note that here it is crucial that $1 \sqsubseteq 2$ etc. don't hold.

Translated to Haskell, monotonicity means that we cannot use $\perp$ as a condition, i.e. we cannot pattern match on $\perp$, or its equivalent `undefined`. Otherwise, the example g from above could be expressed as a Haskell program. As we shall see later, $\perp$ also denotes non-terminating programs, so that the inability to observe $\perp$ inside Haskell is related to the halting problem.

Of course, the notion of *more defined than* can be extended to partial functions by saying that a function is more defined than another if it is so at every possible argument:

$$f \sqsubseteq g \text{ if } \forall x. f(x) \sqsubseteq g(x)$$

Thus, the partial functions also form a poset, with the undefined function $\bot(x) = \bot$ being the least element.

58.3 Recursive Definitions as Fixed Point Iterations

58.3.1 Approximations of the Factorial Function

Now that we have a means to describe partial functions, we can give an interpretation to recursive definitions. Lets take the prominent example of the factorial function $f(n) = n!$ whose recursive definition is

$$f(n) = \text{ if } n == 0 \text{ then } 1 \text{ else } n \cdot f(n-1)$$

Although we saw that interpreting this recursive function directly as a set description may lead to problems, we intuitively know that in order to calculate $f(n)$ for every given n we have to iterate the right hand side. This iteration can be formalized as follows: we calculate a sequence of functions f_k with the property that each one consists of the right hand side applied to the previous one, that is

$$f_{k+1}(n) = \text{ if } n == 0 \text{ then } 1 \text{ else } n \cdot f_k(n-1)$$

We start with the undefined function $f_0(n) = \bot$, and the resulting sequence of partial functions reads:

$$f_1(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ \bot & \text{else} \end{cases}, \; f_2(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ \bot & \text{else} \end{cases}, \; f_3(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ 2 & \text{if } n \text{ is } 2 \\ \bot & \text{else} \end{cases}$$

and so on. Clearly,

$$\bot = f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$$

and we expect that the sequence converges to the factorial function.

The iteration follows the well known scheme of a fixed point iteration

$$x_0, g(x_0), g(g(x_0)), g(g(g(x_0))), \ldots$$

In our case, x_0 is a function and g is a *functional*, a mapping between functions. We have

$$x_0 = \perp$$

and

$$g(x) = n \mapsto \text{ if } n == 0 \text{ then } 1 \text{ else } n * x(n-1)$$

If we start with $x_0 = \perp$, the iteration will yield increasingly defined approximations to the factorial function

$$\perp \sqsubseteq g(\perp) \sqsubseteq g(g(\perp)) \sqsubseteq g(g(g(\perp))) \sqsubseteq \ldots$$

(Proof that the sequence increases: The first inequality $\perp \sqsubseteq g(\perp)$ follows from the fact that $\perp$ is less defined than anything else. The second inequality follows from the first one by applying g to both sides and noting that g is monotone. The third follows from the second in the same fashion and so on.)

It is very illustrative to formulate this iteration scheme in Haskell. As functionals are just ordinary higher order functions, we have

```
g :: (Integer -> Integer) -> (Integer -> Integer)
g x = \n -> if n == 0 then 1 else n * x (n-1)

x0 :: Integer -> Integer
x0 = undefined

(f0:f1:f2:f3:f4:fs) = iterate g x0
```

We can now evaluate the functions `f0,f1,...` at sample arguments and see whether they yield `undefined` or not:

```
> f3 0
1
> f3 1
1
> f3 2
2
> f3 5
*** Exception: Prelude.undefined
> map f3 [0..]
```

```
[1,1,2,*** Exception: Prelude.undefined
> map f4 [0..]
[1,1,2,6,*** Exception: Prelude.undefined
> map f1 [0..]
[1,*** Exception: Prelude.undefined
```

Of course, we cannot use this to check whether f4 is really undefined for all arguments.

58.3.2 Convergence

To the mathematician, the question whether this sequence of approximations converges is still to be answered. For that, we say that a poset is a **directed complete partial order** (**dcpo**) iff every monotone sequence $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ (also called *chain*) has a least upper bound (supremum)

$$\sup_{\sqsubseteq}\{x_0 \sqsubseteq x_1 \sqsubseteq \dots\} = x$$

.

If that's the case for the semantic approximation order, we clearly can be sure that monotone sequence of functions approximating the factorial function indeed has a limit. For our denotational semantics, we will only meet dcpos which have a least element $\perp$ which are called **complete partial order**s (**cpo**).

The `Integer`s clearly form a (d)cpo, because the monotone sequences consisting of more than one element must be of the form

$$\perp \sqsubseteq \cdots \sqsubseteq \ \perp \sqsubseteq n \sqsubseteq n \sqsubseteq \cdots \sqsubseteq n$$

where n is an ordinary number. Thus, n is already the least upper bound.

For functions `Integer -> Integer`, this argument fails because monotone sequences may be of infinite length. But because `Integer` is a (d)cpo, we know that for every point n, there is a least upper bound

$$\sup_{\sqsubseteq}\{\perp = f_0(n) \sqsubseteq f_1(n) \sqsubseteq f_2(n) \sqsubseteq \dots\} =: f(n)$$

.

As the semantic approximation order is defined point-wise, the function f is the supremum we looked for.

These have been the last touches for our aim to transform the impredicative definition of the factorial function into a well defined construction. Of course, it remains to be shown that $f(n)$ actually yields a defined value for every n, but this is not hard and far more reasonable than a completely ill-formed definition.

58.3.3 Bottom includes Non-Termination

It is instructive to try our newly gained insight into recursive definitions on an example that does not terminate:

$$f(n) = f(n+1)$$

The approximating sequence reads

$$f_0 = \perp, f_1 = \perp, \ldots$$

and consists only of $\perp$. Clearly, the resulting limit is $\perp$ again. From an operational point of view, a machine executing this program will loop indefinitely. We thus see that $\perp$ may also denote a **non-terminating** function or value. Hence, given the halting problem, pattern matching on $\perp$ in Haskell is impossible.

58.3.4 Interpretation as Least Fixed Point

Earlier, we called the approximating sequence an example of the well known ”fixed point iteration” scheme. And of course, the definition of the factorial function f can also be thought as the specification of a fixed point of the functional g:

$$f = g(f) = n \mapsto \text{ if } n == 0 \text{ then } 1 \text{ else } n \cdot f(n-1)$$

However, there might be multiple fixed points. For instance, there are several f which fulfill the specification

$$f = n \mapsto \text{ if } n == 0 \text{ then } 1 \text{ else } f(n+1)$$

,

Of course, when executing such a program, the machine will loop forever on $f(1)$ or $f(2)$ and thus not produce any valuable information about the value of $f(1)$. This corresponds to choosing the *least defined* fixed point as semantic object f and this is indeed a canonical choice. Thus, we say that

$$f = g(f)$$

,

defines the **least fixed point** f of g. Clearly, *least* is with respect to our semantic approximation order $\sqsubseteq$.

The existence of a least fixed point is guaranteed by our iterative construction if we add the condition that g must be **continuous** (sometimes also called "chain continuous"). That simply means that g respects suprema of monotone sequences:

$$\sup_{\sqsubseteq}\{g(x_0) \sqsubseteq g(x_1) \sqsubseteq \dots\} = g\left(\sup_{\sqsubseteq}\{x_0 \sqsubseteq x_1 \sqsubseteq \dots\}\right)$$

We can then argue that with

$$f = \sup_{\sqsubseteq}\{x_0 \sqsubseteq g(x_0) \sqsubseteq g(g(x_0)) \sqsubseteq \dots\}$$

we have

$$\begin{array}{rcl} g(f) & = & g\left(\sup_{\sqsubseteq}\{x_0 \sqsubseteq g(x_0) \sqsubseteq g(g(x_0)) \sqsubseteq \dots\}\right) \\ & = & \sup_{\sqsubseteq}\{g(x_0) \sqsubseteq g(g(x_0)) \sqsubseteq \dots\} \\ & = & \sup_{\sqsubseteq}\{x_0 \sqsubseteq g(x_0) \sqsubseteq g(g(x_0)) \sqsubseteq \dots\} \\ & = & f \end{array}$$

and the iteration limit is indeed a fixed point of g. You may also want to convince yourself that the fixed point iteration yields the *least* fixed point possible.

Exercises:
Prove that the fixed point obtained by fixed point iteration starting with $x_0 = \bot$ is also the least one, that it is smaller than any other fixed point. (Hint: $\bot$ is the least element of our cpo and g is monotone)

By the way, how do we know that each Haskell function we write down indeed is continuous? Just as with monotonicity, this has to be enforced by the programming language. Admittedly, these properties can somewhat be enforced or broken at will, so the question feels a bit void. But intuitively, monotonicity is guaranteed by not allowing pattern matches on $\bot$. For continuity, we note that for an arbitrary type `a`, every simple function `a -> Integer` is automatically continuous because the monotone sequences of `Integer`s are of finite length. Any infinite chain of values of type `a` gets mapped to a finite chain of `Integer`s and respect for suprema becomes a consequence of monotonicity. Thus, all functions of the special case `Integer -> Integer` must be continuous. For functionals like g`::(Integer -> Integer) -> (Integer -> Integer)`, the continuity then materializes due to currying, as the type is isomorphic to `::((Integer -> Integer), Integer) -> Integer` and we can take `a=((Integer -> Integer), Integer)`.

In Haskell, the fixed interpretation of the factorial function can be coded as

```
factorial = fix g
```

with the help of the fixed point combinator

$$\texttt{fix :: (a -> a) -> a}.$$

We can define it by

$$\texttt{fix f = let x = f x in x}$$

which leaves us somewhat puzzled because when expanding $factorial$, the result is not anything different from how we would have defined the factorial function in Haskell in the first place. But of course, the construction this whole section was about is not at all present when running a real Haskell program. It's just a means to put the mathematical interpretation of Haskell programs on a firm ground. Yet it is very nice that we can explore these semantics in Haskell itself with the help of `undefined`.

58.4 Strict and Non-Strict Semantics

After having elaborated on the denotational semantics of Haskell programs, we will drop the mathematical function notation $f(n)$ for semantic objects in favor of their now equivalent Haskell notation `f n`.

58.4.1 Strict Functions

A function `f` with one argument is called **strict**, if and only if

$$\texttt{f}\ \bot\ \texttt{=}\ \bot.$$

Here are some examples of strict functions

```
id     x = x
succ   x = x + 1
power2 0 = 1
power2 n = 2 * power2 (n-1)
```

and there is nothing unexpected about them. But why are they strict? It is instructive to prove that these functions are indeed strict. For `id`, this follows from the definition. For `succ`, we have to ponder whether $\bot$ `+ 1`is $\bot$ or not. If it was not, then we should for example have $\bot$ `+ 1 = 2`or more general $\bot$ `+ 1 =` k for some concrete number k. We remember that every function is *monotone*, so we should have for example

$$2 = \bot + 1 \sqsubseteq 4 + 1 = 5$$

as $\bot$ $\sqsubseteq$4. But neither of 2 $\sqsubseteq$ 5, 2 = 5 nor 2 $\sqsupseteq$ 5is valid so that k cannot be 2. In general, we obtain the contradiction

$$k = \bot + 1 \sqsubseteq k + 1 = k + 1.$$

and thus the only possible choice is

$$\texttt{succ}\ \bot = \bot + 1 = \bot$$

and `succ` is strict.

Exercises:
Prove that `power2` is strict. While one can base the proof on the ”obvious” fact that `power2` n is 2^n, the latter is preferably proven using fixed point iteration.

58.4.2 Non-Strict and Strict Languages

Searching for **non-strict** functions, it happens that there is only one prototype of a non-strict function of type `Integer -> Integer`:

```
one x = 1
```

Its variants are `constk x =` k for every concrete number k. Why are these the only ones possible? Remember that `one` n can be no less defined than `one` $\perp$. As `Integer` is a flat domain, both must be equal.

Why is `one` non-strict? To see that it is, we use a Haskell interpreter and try

```
> one (undefined :: Integer)
1
```

which is not $\perp$. This is reasonable as `one` completely ignores its argument. When interpreting $\perp$ in an operational sense as ”non-termination”, one may say that the non-strictness of `one` means that it does not force its argument to be evaluated and therefore avoids the infinite loop when evaluating the argument $\perp$. But one might as well say that every function must evaluate its arguments before computing the result which means that `one` $\perp$ should be $\perp$, too. That is, if the program computing the argument does not halt, `one` should not halt as well.[13] It shows up that one can *choose freely* this or the other design for a functional programming language. One says that the language is *strict* or *non-strict* depending on whether functions are strict or non-strict by default. The choice for Haskell is non-strict. In contrast, the functional languages ML and Lisp choose strict semantics.

58.4.3 Functions with several Arguments

The notion of strictness extends to functions with several variables. For example, a function `f` of two arguments is *strict in the second argument* if and only if

$$\texttt{f x } \perp \texttt{ = } \perp$$

for every `x`. But for multiple arguments, mixed forms where the strictness depends on the given value of the other arguments, are much more common. An example is the conditional

[13] Strictness as premature evaluation of function arguments is elaborated in the chapter Graph Reduction ^{Chapter65 on page 463}.

```
cond b x y = if b then x else y
```

We see that it is strict in `y` depending on whether the test `b` is `True` or `False`:

```
cond True  x ⊥ = x
cond False x ⊥ = ⊥
```

and likewise for `x`. Apparently, `cond` is certainly ⊥ if both `x` and `y` are, but not necessarily when at least one of them is defined. This behavior is called **joint strictness**.

Clearly, `cond` behaves like the if-then-else statement where it is crucial not to evaluate both the `then` and the `else` branches:

```
if null xs then 'a' else head xs
if n == 0  then  1  else 5 / n
```

Here, the else part is ⊥ when the condition is met. Thus, in a non-strict language, we have the possibility to wrap primitive control statements such as if-then-else into functions like `cond`. This way, we can define our own control operators. In a strict language, this is not possible as both branches will be evaluated when calling `cond` which makes it rather useless. This is a glimpse of the general observation that non-strictness offers more flexibility for code reuse than strictness. See the chapter Laziness[14][15] for more on this subject.

58.5 Algebraic Data Types

After treating the motivation case of partial functions between `Integer`s, we now want to extend the scope of denotational semantics to arbitrary algebraic data types in Haskell.

A word about nomenclature: the collection of semantic objects for a particular type is usually called a **domain**. This term is more a generic name than a particular definition and we decide that our domains are cpos (complete partial orders), that is sets of values together with a relation *more defined* that obeys some conditions to allow fixed point iteration. Usually, one adds additional conditions to the cpos that ensure that the values of our domains can be represented in some finite way on a computer and thereby avoiding to ponder the twisted ways of uncountable infinite sets. But as we are not going to prove general domain theoretic theorems, the conditions will just happen to hold by construction.

58.5.1 Constructors

Let's take the example types

[14] Chapter 66 on page 475

[15] The term *Laziness* comes from the fact that the prevalent implementation technique for non-strict languages is called *lazy evaluation*

```
data Bool    = True | False
data Maybe a = Just a | Nothing
```

Here, `True`, `False` and `Nothing` are nullary constructors whereas `Just` is a unary constructor. The inhabitants of `Bool` form the following domain:

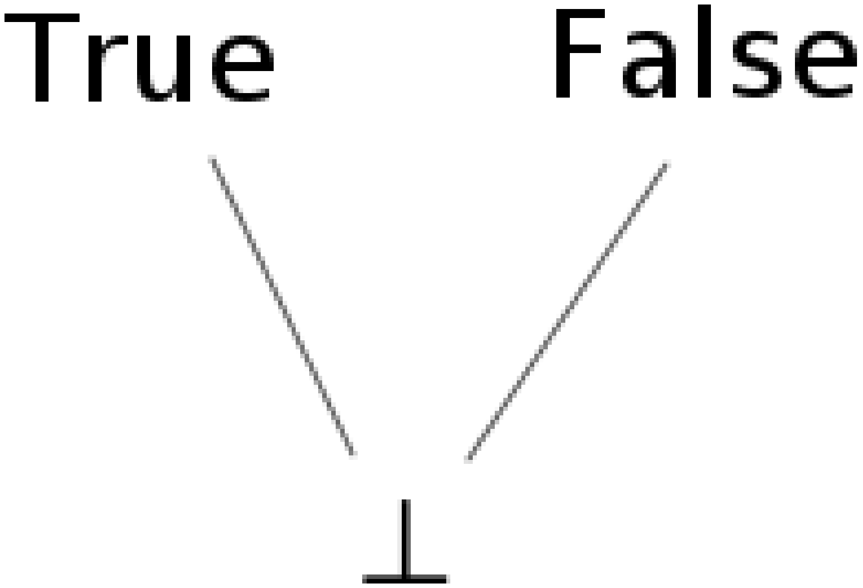


Figure 25

Remember that ⊥ is added as least element to the set of values `True` and `False`, we say that the type is **lifted**[16]. A domain whose poset diagram consists of only one level is called a **flat domain**. We already know that *Integer* is a flat domain as well, it's just that the level above ⊥ has an infinite number of elements.

What are the possible inhabitants of `Maybe Bool`? They are

```
⊥, Nothing, Just ⊥, Just True, Just False
```

So the general rule is to insert all possible values into the unary (binary, ternary, ...) constructors as usual but without forgetting ⊥. Concerning the partial order, we remember the condition that the constructors should be monotone just as any other functions. Hence, the partial order looks as follows

[16] The term *lifted* is somewhat overloaded, see also Unboxed Types ^{Chapter58.1.2 on page 407}.

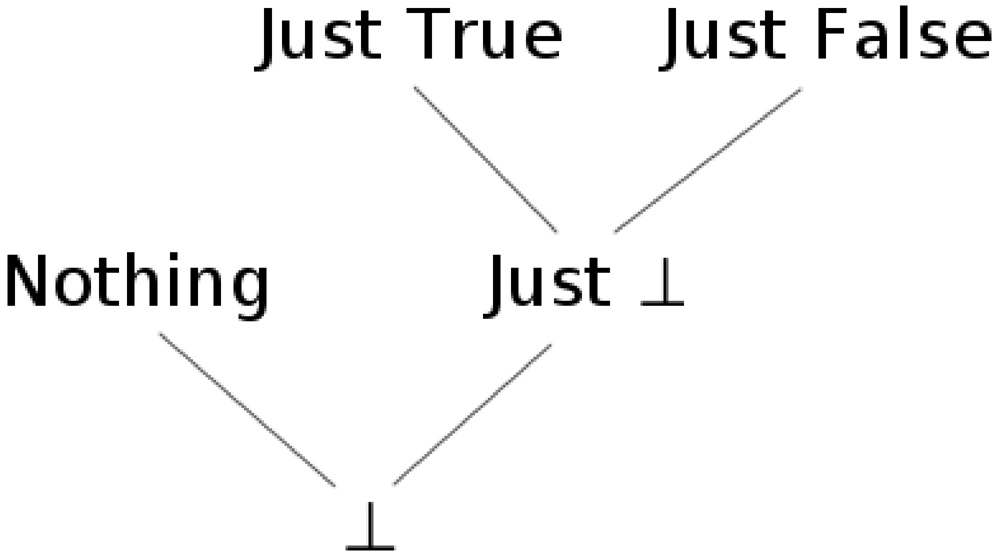


Figure 26

But there is something to ponder: why isn't `Just ⊥ = ⊥`? I mean "Just undefined" is as undefined as "undefined"! The answer is that this depends on whether the language is strict or non-strict. In a strict language, all constructors are strict by default, i.e. `Just ⊥ = ⊥` and the diagram would reduce to

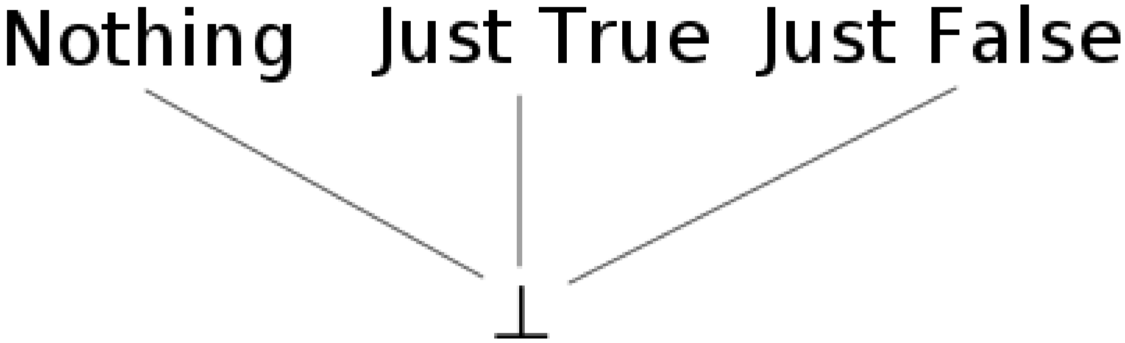


Figure 27

As a consequence, all domains of a strict language are flat.

But in a non-strict language like Haskell, constructors are non-strict by default and `Just ⊥` is a new element different from ⊥, because we can write a function that reacts differently to them:

```
f (Just _) = 4
f Nothing  = 7
```

As `f` ignores the contents of the `Just` constructor, `f (Just ⊥)` is `4` but `f ⊥` is ⊥ (intuitively, if `f` is passed ⊥, it will not be possible to tell whether to take the Just branch or the Nothing branch, and so ⊥ will be returned).

This gives rise to **non-flat domains** as depicted in the former graph. What should these be of use for? In the context of Graph Reduction[17], we may also think of ⊥ as an unevaluated expression. Thus, a value `x = Just ⊥` may tell us that a computation (say a lookup) succeeded and is not `Nothing`, but that the true value has not been evaluated yet. If we are only interested in whether `x` succeeded or not, this actually saves us from the unnecessary work to calculate whether `x` is `Just True` or `Just False` as would be the case in a flat domain. The full impact of non-flat domains will be explored in the chapter Laziness[18], but one prominent example are infinite lists treated in section Recursive Data Types and Infinite Lists[19].

58.5.2 Pattern Matching

In the section Strict Functions[20], we proved that some functions are strict by inspecting their results on different inputs and insisting on monotonicity. However, in the light of algebraic data types, there can only be one source of strictness in real life Haskell: pattern matching, i.e. `case` expressions. The general rule is that pattern matching on a constructor of a `data`-type will force the function to be strict, i.e. matching ⊥ against a constructor always gives ⊥. For illustration, consider

```
const1 _ = 1
```

```
const1' True  = 1
const1' False = 1
```

The first function `const1` is non-strict whereas the `const1'` is strict because it decides whether the argument is `True` or `False` although its result doesn't depend on that. Pattern matching in function arguments is equivalent to `case`-expressions

```
const1' x = case x of
   True  -> 1
   False -> 1
```

which similarly impose strictness on `x`: if the argument to the `case` expression denotes ⊥ the whole `case` will denote ⊥, too. However, the argument for case expressions may be more involved as in

[17] Chapter 65 on page 463
[18] Chapter 66 on page 475
[19] Chapter 58.5.3 on page 424
[20] Chapter 58.4.1 on page 417

```
foo k table = case lookup ("Foo." ++ k) table of
  Nothing -> ...
  Just x  -> ...
```

and it can be difficult to track what this means for the strictness of `foo`.

An example for multiple pattern matches in the equational style is the logical `or`:

```
or True _ = True
or _ True = True
or _ _    = False
```

Note that equations are matched from top to bottom. The first equation for `or` matches the first argument against `True`, so `or` is strict in its first argument. The same equation also tells us that `or True x` is non-strict in `x`. If the first argument is `False`, then the second will be matched against `True` and `or False x` is strict in `x`. Note that while wildcards are a general sign of non-strictness, this depends on their position with respect to the pattern matches against constructors.

Exercises:

1. Give an equivalent discussion for the logical `and`
2. Can the logical ”excluded or” (`xor`) be non-strict in one of its arguments if we know the other?

There is another form of pattern matching, namely **irrefutable patterns** marked with a tilde `~`. Their use is demonstrated by

```
f ~(Just x) = 1
f Nothing   = 2
```

An irrefutable pattern always succeeds (hence the name) resulting in `f ⊥ = 1`. But when changing the definition of `f` to

```
f ~(Just x) = x + 1
f Nothing   = 2      -- this line may as well be left away
```

we have

```
f ⊥        = ⊥ + 1 = ⊥
f (Just 1) = 1 + 1 = 2
```

If the argument matches the pattern, `x` will be bound to the corresponding value. Otherwise, any variable like `x` will be bound to ⊥.

By default, `let` and `where` bindings are non-strict, too:

```
foo key map = let Just x = lookup key map in ...
```

is equivalent to

```
foo key map = case (lookup key map) of ~(Just x) -> ...
```

Exercises:

1. The Haskell language definition[a] gives the detailed semantics of pattern matching[b] and you should now be able to understand it. So go on and have a look!
2. Consider a function `or` of two `Bool`ean arguments with the following properties:

```
or ⊥     ⊥    = ⊥
or True  ⊥    = True
or ⊥     True = True

or False y    = y
or x False    = x
```

This function is another example of joint strictness, but a much sharper one: the result is only ⊥ if both arguments are (at least when we restrict the arguments to `True` and ⊥). Can such a function be implemented in Haskell?

[a] http://www.haskell.org/onlinereport/

[b] http://www.haskell.org/onlinereport/exps.html#case-semantics

58.5.3 Recursive Data Types and Infinite Lists

The case of recursive data structures is not very different from the base case. Consider a list of unit values

```
data List = [] | () : List
```

Though this seems like a simple type, there is a surprisingly complicated number of ways you can fit ⊥ in here and there, and therefore the corresponding graph is complicated. The bottom of this graph is shown below. An ellipsis indicates that the graph continues along this direction. A red ellipse behind an element indicates that this is the end of a chain; the element is in normal form.

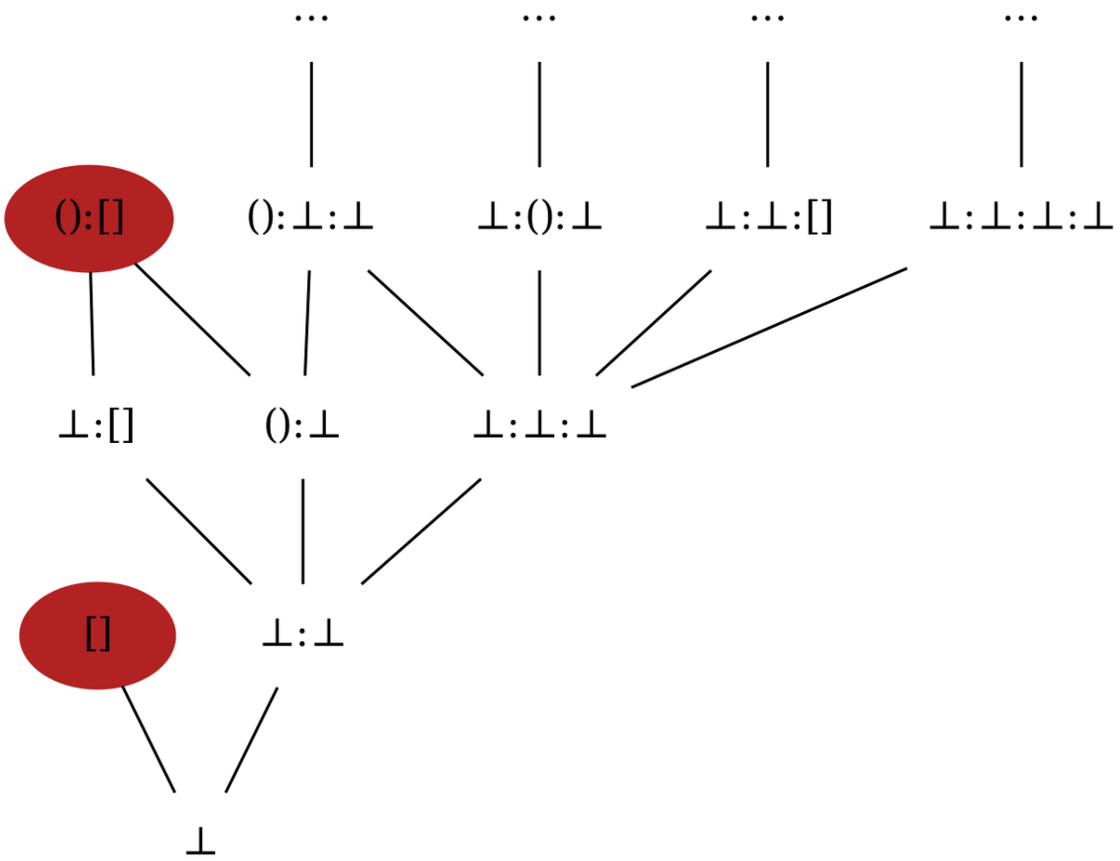


Figure 28

and so on. But now, there are also chains of infinite length like

$$\perp \sqsubseteq \texttt{():}\perp \sqsubseteq \texttt{():():}\perp \sqsubseteq \ldots$$

This causes us some trouble as we noted in section Convergence[21] that every monotone sequence must have a least upper bound. This is only possible if we allow for **infinite lists**. Infinite lists (sometimes also called *streams*) turn out to be very useful and their manifold use cases are treated in full detail in chapter Laziness[22]. Here, we will show what their denotational semantics should be and how to reason about them. Note that while the following discussion is restricted to lists only, it easily generalizes to arbitrary recursive data structures like trees.

In the following, we will switch back to the standard list type

```
data [a] = [] | a : [a]
```

[21] Chapter 58.3.2 on page 414

[22] Chapter 66 on page 475

to close the syntactic gap to practical programming with infinite lists in Haskell.

Exercises:

1. Draw the non-flat domain corresponding `[Bool]`.
2. How is the graphic to be changed for `[Integer]`?

Calculating with infinite lists is best shown by example. For that, we need an infinite list

```
ones :: [Integer]
ones = 1 : ones
```

When applying the fixed point iteration to this recursive definition, we see that `ones` ought to be the supremum of

$$\bot \sqsubseteq \texttt{1:}\bot \sqsubseteq \texttt{1:1:}\bot \sqsubseteq \texttt{1:1:1:}\bot \sqsubseteq \ldots,$$

that is an infinite list of `1`. Let's try to understand what `take 2 ones` should be. With the definition of `take`

```
take 0 _      = []
take n (x:xs) = x : take (n-1) xs
take n []     = []
```

we can apply `take` to elements of the approximating sequence of `ones`:

```
take 2 ⊥       ==>  ⊥
take 2 (1:⊥)   ==>  1 : take 1 ⊥      ==>  1 : ⊥
take 2 (1:1:⊥) ==>  1 : take 1 (1:⊥)  ==>  1 : 1 : take 0 ⊥
               ==>  1 : 1 : []
```

We see that `take 2 (1:1:1:⊥)` and so on must be the same as `take 2 (1:1:⊥) = 1:1:[]` because `1:1:[]` is fully defined. Taking the supremum on both the sequence of input lists and the resulting sequence of output lists, we can conclude

```
take 2 ones = 1:1:[]
```

Thus, taking the first two elements of `ones` behaves exactly as expected.

Generalizing from the example, we see that reasoning about infinite lists involves considering the approximating sequence and passing to the supremum, the truly infinite list. Still, we did not give it a firm ground. The solution is to identify the infinite list with the whole chain itself and to formally add it as a new element to our domain: the infinite list *is* the sequence of its approximations. Of course, any infinite list like `ones` can be compactly depicted as

```
ones = 1 : 1 : 1 : 1 : ...
```

what simply means that

```
ones = (⊥ ⊑ 1:⊥ ⊑ 1:1:⊥ ⊑ ...)
```

Exercises:

1. Of course, there are more interesting infinite lists than `ones`. Can you write recursive definition in Haskell for
   a) the natural numbers `nats = 1:2:3:4:...`
   b) a cycle like `cycle123 = 1:2:3: 1:2:3 : ...`
2. Look at the Prelude functions `repeat` and `iterate` and try to solve the previous exercise with their help.
3. Use the example from the text to find the value the expression `drop 3 nats` denotes.
4. Assume that the work in a strict setting, i.e. that the domain of `[Integer]` is flat. What does the domain look like? What about infinite lists? What value does `ones` denote?

What about the puzzle of how a computer can calculate with infinite lists? It takes an infinite amount of time, after all? Well, this is true. But the trick is that the computer may well finish in a finite amount of time if it only considers a finite part of the infinite list. So, infinite lists should be thought of as *potentially* infinite lists. In general, intermediate results take the form of infinite lists whereas the final value is finite. It is one of the benefits of denotational semantics that one treat the intermediate infinite data structures as truly infinite when reasoning about program correctness.

Exercises:

1. To demonstrate the use of infinite lists as intermediate results, show that
   `take 3 (map (+1) nats) = take 3 (tail nats)`
   by first calculating the infinite sequence corresponding to `map (+1) nats`.
2. Of course, we should give an example where the final result indeed takes an infinite time. So, what does

   ```
   filter (< 5) nats
   ```

   denote?
3. Sometimes, one can replace `filter` with `takeWhile` in the previous exercise. Why only sometimes and what happens if one does?

As a last note, the construction of a recursive domain can be done by a fixed point iteration similar to recursive definition for functions. Yet, the problem of infinite chains has to be tackled explicitly. See the literature in External Links[23] for a formal construction.

[23] Chapter 58.7 on page 431

58.5.4 Haskell specialities: Strictness Annotations and Newtypes

Haskell offers a way to change the default non-strict behavior of data type constructors by *strictness annotations*. In a data declaration like

```
data Maybe' a = Just' !a | Nothing'
```

an exclamation point `!` before an argument of the constructor specifies that it should be strict in this argument. Hence we have `Just' ⊥ = ⊥` in our example. Further information may be found in chapter Strictness[24].

In some cases, one wants to rename a data type, like in

```
data Couldbe a = Couldbe (Maybe a)
```

However, `Couldbe a` contains both the elements ⊥ and `Couldbe ⊥`. With the help of a `newtype` definition

```
newtype Couldbe a = Couldbe (Maybe a)
```

we can arrange that `Couldbe a` is semantically equal to `Maybe a`, but different during type checking. In particular, the constructor `Couldbe` is strict. Yet, this definition is subtly different from

```
data Couldbe' a = Couldbe' !(Maybe a)
```

To explain how, consider the functions

```
f  (Couldbe  m) = 42
f' (Couldbe' m) = 42
```

Here, `f' ⊥` will cause the pattern match on the constructor `Couldbe'` fail with the effect that `f' ⊥ = ⊥`. But for the newtype, the match on `Couldbe` will never fail, we get `f ⊥ = 42`. In a sense, the difference can be stated as:

- for the strict case, `Couldbe' ⊥` is a synonym for ⊥
- for the newtype, ⊥ is a synonym for `Couldbe ⊥`

with the agreement that a pattern match on ⊥ fails and that a match on *`Constructor`* ⊥does not.

Newtypes may also be used to define recursive types. An example is the alternate definition of the list type `[a]`

[24] Chapter 67 on page 487

```
newtype List a = In (Maybe (a, List a))
```

Again, the point is that the constructor `In` does not introduce an additional lifting with ⊥.

58.6 Other Selected Topics

58.6.1 Abstract Interpretation and Strictness Analysis

As lazy evaluation means a constant computational overhead, a Haskell compiler may want to discover where inherent non-strictness is not needed at all which allows it to drop the overhead at these particular places. To that extent, the compiler performs **strictness analysis** just like we proved in some functions to be strict section Strict Functions[25]. Of course, details of strictness depending on the exact values of arguments like in our example `cond` are out of scope (this is in general undecidable). But the compiler may try to find approximate strictness information and this works in many common cases like `power2`.

Now, **abstract interpretation** is a formidable idea to reason about strictness: ...

For more about strictness analysis, see the research papers about strictness analysis on the Haskell wiki[26].

58.6.2 Interpretation as Powersets

So far, we have introduced ⊥ and the semantic approximation order ⊑ abstractly by specifying their properties. However, both as well as any inhabitants of a data type like `Just ⊥` can be interpreted as ordinary sets. This is called the **powerset construction**. NOTE: *i'm not sure whether this is really true. Someone how knows, please correct this.*

The idea is to think of ⊥ as the *set of all possible values* and that a computation retrieves more information this by choosing a subset. In a sense, the denotation of a value starts its life as the set of all values which will be reduced by computations until there remains a set with a single element only.

As an example, consider `Bool` where the domain looks like

```
{True}  {False}
   \      /
    \    /
   ⊥ = {True, False}
```

The values `True` and `False` are encoded as the singleton sets `{True}` and `{False}` and ⊥ is the set of all possible values.

[25] Chapter 58.4.1 on page 417

[26] http://haskell.org/haskellwiki/Research_papers/Compilation#Strictness

Another example is `Maybe Bool`:

```
 {Just True}   {Just False}
        \     /
         \   /
{Nothing} {Just True, Just False}
     \      /
      \    /
 ⊥ = {Nothing, Just True, Just False}
```

We see that the semantic approximation order is equivalent to set inclusion, but with arguments switched:

$$x \sqsubseteq y \Longleftrightarrow x \supseteq y$$

This approach can be used to give a semantics to exceptions in Haskell[27].

58.6.3 Naïve Sets are unsuited for Recursive Data Types

In the section What to choose as Semantic Domain?[28], we argued that taking simple sets as denotation for types doesn't work well with partial functions. In the light of recursive data types, things become even worse as John C. Reynolds showed in his paper *Polymorphism is not set-theoretic*[29].

Reynolds actually considers the recursive type

```
newtype U = In ((U -> Bool) -> Bool)
```

Interpreting `Bool` as the set `{True,False}` and the function type `A -> B` as the set of functions from `A` to `B`, the type `U` cannot denote a set. This is because `(A -> Bool)` is the set of subsets (powerset) of `A` which, due to a diagonal argument analogous to Cantor's argument that there are "more" real numbers than natural ones, always has a bigger cardinality than `A`. Thus, `(U -> Bool) -> Bool` has an even bigger cardinality than `U` and there is no way for it to be isomorphic to `U`. Hence, the set `U` must not exist, a contradiction.

In our world of partial functions, this argument fails. Here, an element of `U` is given by a sequence of approximations taken from the sequence of domains

`⊥, (⊥ -> Bool) -> Bool, (((⊥ -> Bool) -> Bool) -> Bool) -> Bool` and so on

where ⊥ denotes the domain with the single inhabitant ⊥. While the author of this text admittedly has no clue on what such a thing should mean, the constructor gives a perfectly

[27] S. Peyton Jones, A. Reid, T. Hoare, S. Marlow, and F. Henderson. A semantics for imprecise exceptions. ^{http://research.microsoft.com/~simonpj/Papers/imprecise-exn.htm} In Programming Languages Design and Implementation. ACM press, May 1999.

[28] Chapter 58.1.2 on page 407

[29] John C. Reynolds. *Polymorphism is not set-theoretic*. INRIA Rapports de Recherche No. 296. May 1984.

well defined object for `U`. We see that the type `(U -> Bool) -> Bool` merely consists of shifted approximating sequences which means that it is isomorphic to `U`.

As a last note, Reynolds actually constructs an equivalent of `U` in the second order polymorphic lambda calculus. There, it happens that all terms have a normal form, i.e. there are only total functions when we do not include a primitive recursion operator `fix :: (a -> a) -> a`. Thus, there is no true need for partial functions and $\perp$, yet a naïve set theoretic semantics fails. We can only speculate that this has to do with the fact that not every mathematical function is computable. In particular, the set of computable functions `A -> Bool` should not have a bigger cardinality than `A`.

58.7 External Links

w:Denotational semantics[30]

Online books about Denotational Semantics

- Denotational Semantics. A Methodology for Language Development . Allyn and Bacon , , 1986

[30] https://en.wikipedia.org/wiki/Denotational%20semantics

59 Category theory

This article attempts to give an overview of category theory, in so far as it applies to Haskell. To this end, Haskell code will be given alongside the mathematical definitions. Absolute rigour is not followed; in its place, we seek to give the reader an intuitive feel for what the concepts of category theory are and how they relate to Haskell.

59.1 Introduction to categories

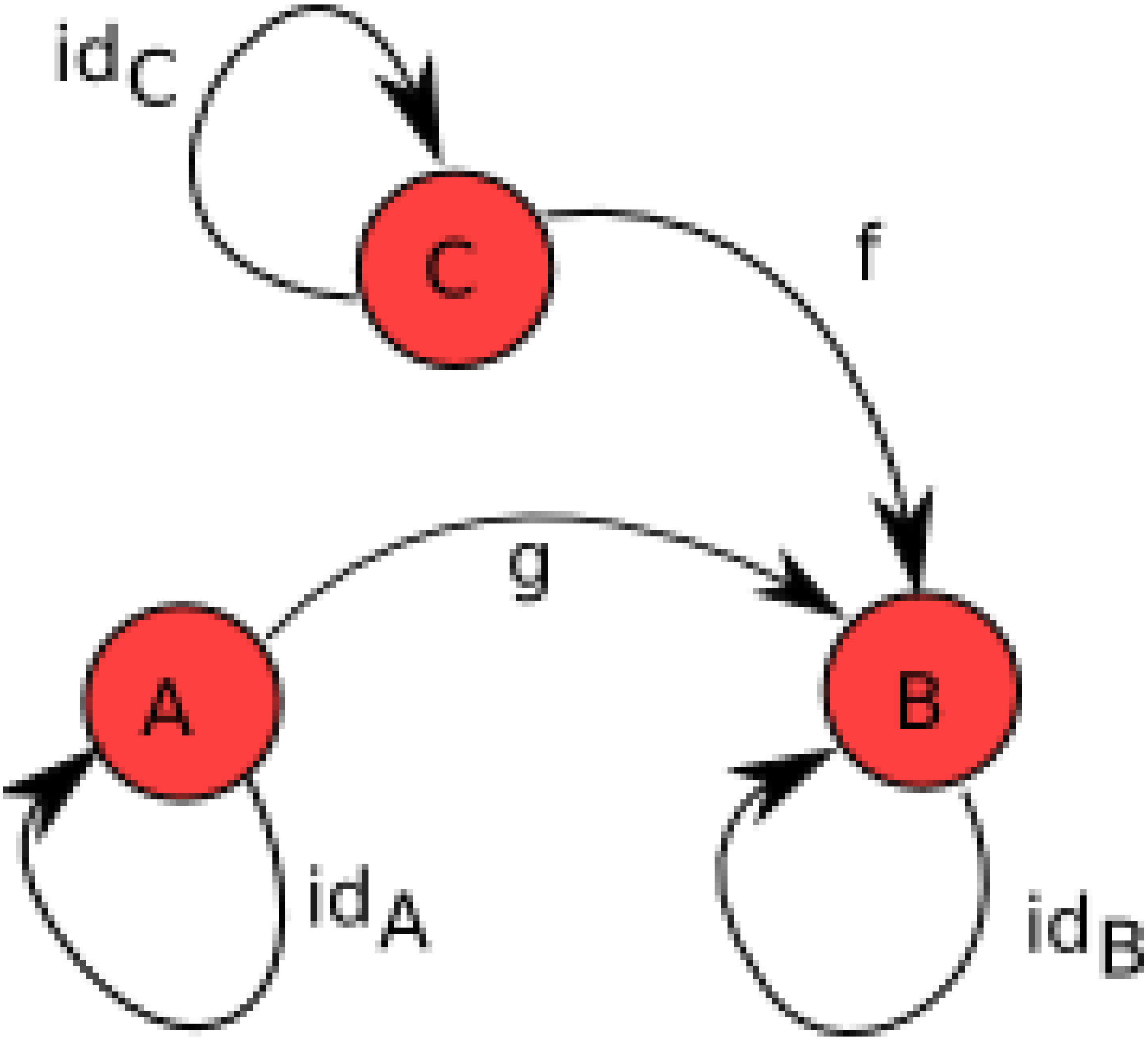


Figure 29 A simple category, with three objects A, B and C, three identity morphisms id_A, id_B and id_C, and two other morphisms $f : C \to B$ and $g : A \to B$. The third element (the specification of how to compose the morphisms) is not shown.

A category is, in essence, a simple collection. It has three components:

- A collection of **objects**.
- A collection of **morphisms**, each of which ties two objects (a *source object* and a *target object*) together. (These are sometimes called **arrows**, but we avoid that term here as it has other connotations in Haskell.) If f is a morphism with source object A and target object B, we write $f : A \to B$.
- A notion of **composition** of these morphisms. If $g : A \to B$ and $f : B \to C$ are two morphisms, they can be composed, resulting in a morphism $f \circ g : A \to C$.

Lots of things form categories. For example, **Set** is the category of all sets with morphisms as standard functions and composition being standard function composition. (Category names are often typeset in bold face.) **Grp** is the category of all groups with morphisms as functions that preserve group operations (the group homomorphisms), i.e. for any two groups, G with operation *** and H with operation $\cdot$, a function $f : G \to H$ is a morphism in **Grp** if:

$$f(u * v) = f(u) \cdot f(v)$$

It may seem that morphisms are always functions, but this needn't be the case. For example, any partial order $(P, \leq)$ defines a category where the objects are the elements of P, and there is a morphism between any two objects A and B iff $A \leq B$. Moreover, there are allowed to be multiple morphisms with the same source and target objects; using the **Set** example, sin and cos are both functions with source object $\mathbb{R}$ and target object $[-1,1]$, but they're most certainly not the same morphism!

59.1.1 Category laws

There are three laws that categories need to follow. Firstly, and most simply, the composition of morphisms needs to be ***associative***. Symbolically,

$$f \circ (g \circ h) = (f \circ g) \circ h$$

Morphisms are applied right to left in Haskell and most commonly in mathematics, so with $f \circ g$ first g is applied, then f.

Secondly, the category needs to be ***closed*** under the composition operation. So if $f : B \to C$ and $g : A \to B$, then there must be some morphism $h : A \to C$ in the category such that $h = f \circ g$. We can see how this works using the following category:

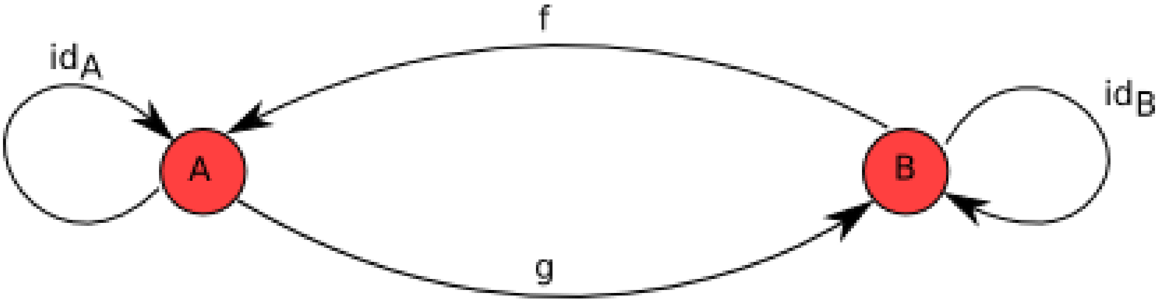


Figure 30

f and g are both morphisms so we must be able to compose them and get another morphism in the category. So which is the morphism $f \circ g$? The only option is id_A. Similarly, we see that $g \circ f = id_B$.

Lastly, given a category C there needs to be for every object `A` an ***identity*** *morphism*, $id_A : A \to A$ that is an identity of composition with other morphisms. Put precisely, for every morphism $g : A \to B$:

$$g \circ id_A = id_B \circ g = g$$

59.1.2 **Hask**, the Haskell category

The main category we'll be concerning ourselves with in this article is **Hask**, which treats Haskell types as objects and Haskell functions as morphisms and uses `(.)` for composition: a function `f :: A -> B` for types `A` and `B` is a morphism in **Hask**. We can check the first and second law easily: we know `(.)` is an associative function, and clearly, for any `f` and `g`, `f . g` is another function. In **Hask**, the identity morphism is `id`, and we have trivially:

```
id . f = f . id = f
```

[1] This isn't an exact translation of the law above, though; we're missing subscripts. The function `id` in Haskell is *polymorphic*—it can take many different types for its domain and range, or, in category-speak, can have many different source and target objects. But morphisms in category theory are by definition *monomorphic*—each morphism has one specific source object and one specific target object. A polymorphic Haskell function can be made monomorphic by specifying its type (*instantiating* with a monomorphic type), so it would be more precise if we said that the identity morphism from **Hask** on a type `A` is `(id :: A -> A)`. With this in mind, the above law would be rewritten as:

```
(id :: B -> B) . f = f . (id :: A -> A) = f
```

[1] Actually, there is a subtlety here: because `(.)` is a lazy function, if `f` is `undefined`, we have that `id . f = \_ -> ⊥`. Now, while this may seem equivalent to ⊥ for all intents and purposes, you can actually tell them apart using the strictifying function `seq`, meaning that the last category law is broken. We can define a new strict composition function, `f .! g = ((.) $! f) $! g`, that makes **Hask** a category. We proceed by using the normal `(.)`, though, and attribute any discrepancies to the fact that `seq` breaks an awful lot of the nice language properties anyway.

However, for simplicity, we will ignore this distinction when the meaning is clear.

Exercises:

- As was mentioned, any partial order $(P, \leq)$ is a category with objects as the elements of P and a morphism between elements a and b iff a $\leq$ b. Which of the above laws guarantees the transitivity of $\leq$?
- (Harder.) If we add another morphism to the above example, it fails to be a category. Why? Hint: think about associativity of the composition operation.

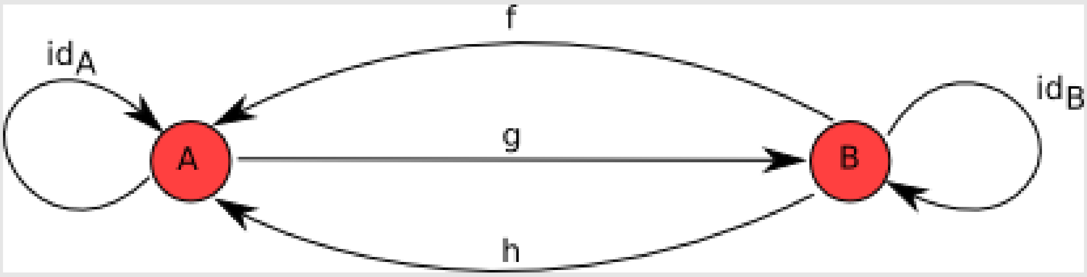


Figure 31

59.2 Functors

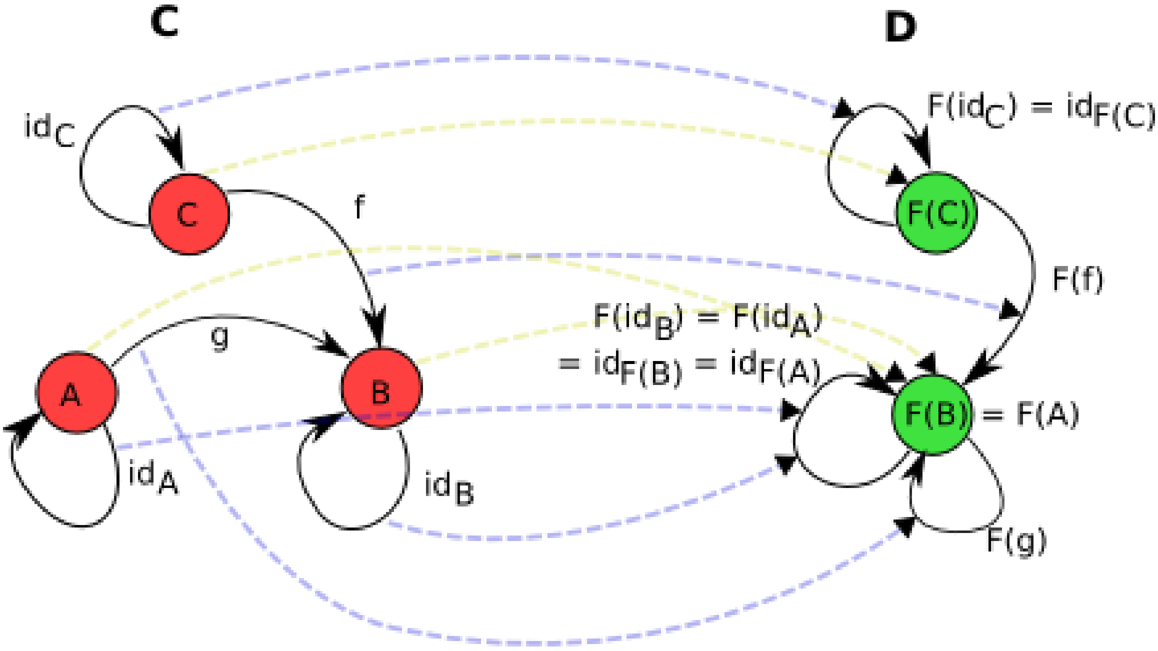


Figure 32 A functor between two categories, **C** and **D**. Of note is that the objects A and B both get mapped to the same object in **D**, and that therefore g becomes a morphism with the same source and target object (but isn't necessarily an identity), and id_A and id_B become the same morphism. The arrows showing the mapping of objects are shown in a dotted, pale olive. The arrows showing the mapping of morphisms are shown in a dotted, pale blue.

So we have some categories which have objects and morphisms that relate our objects together. The next Big Topic in category theory is the **functor**, which relates categories together. A functor is essentially a transformation between categories, so given categories C and D, a functor $F : C \to D$:

- Maps any object A in C to $F(A)$, in D.
- Maps morphisms $f : A \to B$ in C to $F(f) : F(A) \to F(B)$ in D.

One of the canonical examples of a functor is the forgetful functor **Grp** → **Set** which maps groups to their underlying sets and group morphisms to the functions which behave the same but are defined on sets instead of groups. Another example is the power set functor **Set** → **Set** which maps sets to their power sets and maps functions $f : X \to Y$ to functions $\mathcal{P}(X) \to \mathcal{P}(Y)$ which take inputs $U \subseteq X$ and return $f(U)$, the image of U under f, defined by $f(U) = \{\, f(u) : u \in U\,\}$. For any category C, we can define a functor known as the identity functor on C, or $1_C : C \to C$, that just maps objects to themselves and morphisms to themselves. This will turn out to be useful in the monad laws[2] section later on.

Once again there are a few axioms that functors have to obey. Firstly, given an identity morphism id_A on an object A, $F(id_A)$ must be the identity morphism on $F(A)$, i.e.:

$$F(id_A) = id_{F(A)}$$

Secondly functors must distribute over morphism composition, i.e.

$$F(f \circ g) = F(f) \circ F(g)$$

Exercises:
For the diagram given on the right, check these functor laws.

59.2.1 Functors on Hask

The Functor typeclass you have probably seen in Haskell does in fact tie in with the categorical notion of a functor. Remember that a functor has two parts: it maps objects in one category to objects in another and morphisms in the first category to morphisms in the second. Functors in Haskell are from **Hask** to *func*, where *func* is the subcategory of **Hask** defined on just that functor's types. E.g. the list functor goes from **Hask** to **Lst**, where **Lst** is the category containing only *list types*, that is, `[T]` for any type `T`. The morphisms in **Lst** are functions defined on list types, that is, functions `[T] -> [U]` for types `T`, `U`. How does this tie into the Haskell typeclass Functor? Recall its definition:

```
class Functor (f :: * -> *) where
  fmap :: (a -> b) -> f a -> f b
```

[2] Chapter 59.4 on page 441

Let's have a sample instance, too:

```
instance Functor Maybe where
  fmap f (Just x) = Just (f x)
  fmap _ Nothing  = Nothing
```

Here's the key part: the *type constructor* Maybe takes any type `T` to a new type, `Maybe T`. Also, `fmap` restricted to Maybe types takes a function `a -> b` to a function `Maybe a -> Maybe b`. But that's it! We've defined two parts, something that takes objects in **Hask** to objects in another category (that of Maybe types and functions defined on Maybe types), and something that takes morphisms in **Hask** to morphisms in this category. So Maybe is a functor.

A useful intuition regarding Haskell functors is that they represent types that can be mapped over. This could be a list or a Maybe, but also more complicated structures like trees. A function that does some mapping could be written using `fmap`, then any functor structure could be passed into this function. E.g. you could write a generic function that covers all of Data.List.map, Data.Map.map, Data.Array.IArray.amap, and so on.

What about the functor axioms? The polymorphic function `id` takes the place of id_A for any A, so the first law states:

```
fmap id = id
```

With our above intuition in mind, this states that mapping over a structure doing nothing to each element is equivalent to doing nothing overall. Secondly, morphism composition is just `(.)`, so

```
fmap (f . g) = fmap f . fmap g
```

This second law is very useful in practice. Picturing the functor as a list or similar container, the right-hand side is a two-pass algorithm: we map over the structure, performing `g`, then map over it again, performing `f`. The functor axioms guarantee we can transform this into a single-pass algorithm that performs `f . g`. This is a process known as *fusion*.

Exercises:
Check the laws for the Maybe and list functors.

59.2.2 Translating categorical concepts into Haskell

Functors provide a good example of how category theory gets translated into Haskell. The key points to remember are that:

- We work in the category **Hask** and its subcategories.
- Objects are types.
- Morphisms are functions.
- Things that take a type and return another type are type constructors.

- Things that take a function and return another function are higher-order functions.
- Typeclasses, along with the polymorphism they provide, make a nice way of capturing the fact that in category theory things are often defined over a number of objects at once.

59.3 Monads

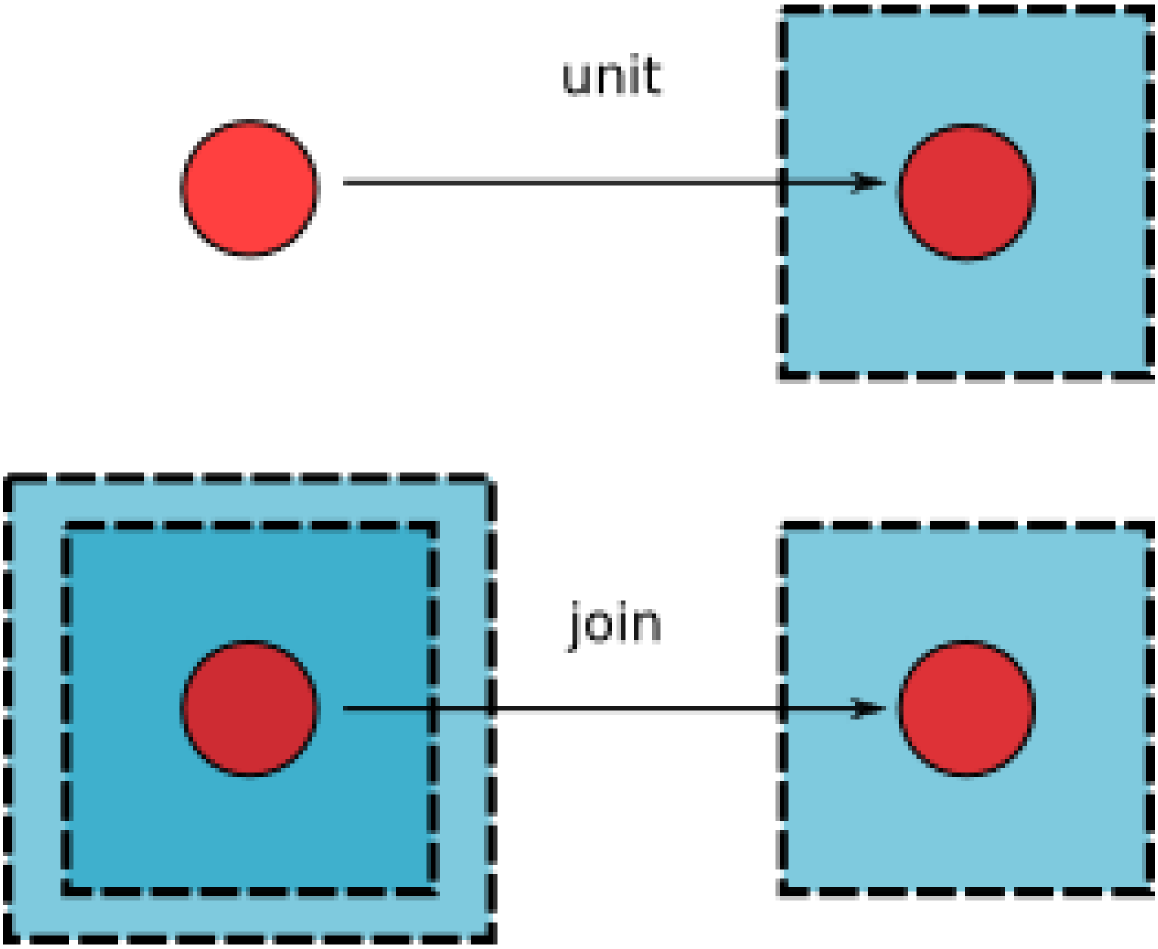


Figure 33 *unit* and *join*, the two morphisms that must exist for every object for a given monad.

Monads are obviously an extremely important concept in Haskell, and in fact they originally came from category theory. A *monad* is a special type of functor, from a category to that same category, that supports some additional structure. So, down to definitions. A monad is a functor $M : C \to C$, along with two morphisms[3] for every object X in C:

[3] Experienced category theorists will notice that we're simplifying things a bit here; instead of presenting *unit* and *join* as natural transformations, we treat them explicitly as morphisms, and require naturality as extra axioms alongside the standard monad laws (laws 3 and 4) ^{Chapter59.4.3 on page 444}. The reasoning is simplicity; we are not trying to teach category theory as a whole, simply give a categorical background to some of the structures in Haskell. You may also notice that we are giving these morphisms names suggestive of their Haskell analogues, because the names η and μ don't provide much intuition.

- $unit^M_X : X \to M(X)$
- $join^M_X : M(M(X)) \to M(X)$

When the monad under discussion is obvious, we'll leave out the M superscript for these functions and just talk about $unit_X$ and $join_X$ for some X.

Let's see how this translates to the Haskell typeclass Monad, then.

```
class Functor m => Monad m where
  return :: a -> m a
  (>>=)  :: m a -> (a -> m b) -> m b
```

The class constraint of `Functor m` ensures that we already have the functor structure: a mapping of objects and of morphisms. `return` is the (polymorphic) analogue to $unit_X$ for any X. But we have a problem. Although `return`'s type looks quite similar to that of *unit*; the other function, `(>>=)`, often called *bind*, bears no resemblance to *join*. There is however another monad function, `join :: Monad m => m (m a) -> m a`, that looks quite similar. Indeed, we can recover `join` and `(>>=)` from each other:

```
join :: Monad m => m (m a) -> m a
join x = x >>= id

(>>=) :: Monad m => m a -> (a -> m b) -> m b
x >>= f = join (fmap f x)
```

So specifying a monad's `return`, `fmap`, and `join` is equivalent to specifying its `return` and `(>>=)`. It just turns out that the normal way of defining a monad in category theory is to give *unit* and *join*, whereas Haskell programmers like to give `return` and `(>>=)`.[4] Often, the categorical way makes more sense. Any time you have some kind of structure M and a natural way of taking any object X into $M(X)$, as well as a way of taking $M(M(X))$ into $M(X)$, you probably have a monad. We can see this in the following example section.

59.3.1 Example: the powerset functor is also a monad

The power set functor $P : \mathbf{Set} \to \mathbf{Set}$ described above forms a monad. For any set S you have a $unit_S(x) = \{x\}$, mapping elements to their singleton set. Note that each of these singleton sets are trivially a subset of S, so $unit_S$ returns elements of the powerset of S, as is required. Also, you can define a function $join_S$ as follows: we receive an input $L \in \mathcal{P}(\mathcal{P}(S))$. This is:

- A member of the powerset of the powerset of S.
- So a member of the set of all subsets of the set of all subsets of S.
- So a set of subsets of S

[4] This is perhaps due to the fact that Haskell programmers like to think of monads as a way of sequencing computations with a common feature, whereas in category theory the container aspect of the various structures is emphasised. `join` pertains naturally to containers (squashing two layers of a container down into one), but `(>>=)` is the natural sequencing operation (do something, feeding its results into something else).

We then return the union of these subsets, giving another subset of S. Symbolically,

$$join_S(L) = \bigcup L$$

Hence P is a monad [5].

In fact, P is almost equivalent to the list monad; with the exception that we're talking lists instead of sets, they're almost the same. Compare:

Power set functor on Set	
Function type	**Definition**
Given a set S and a morphism $f : A \to B$:	
$P(f) : \mathcal{P}(A) \to \mathcal{P}(B)$	$(P(f))(S) = \{f(a) : a \in S\}$
$unit_S : S \to \mathcal{P}(S)$	$unit_S(x) = \{x\}$
$join_S : \mathcal{P}(\mathcal{P}(S)) \to \mathcal{P}(S)$	$join_S(L) = \bigcup L$
List monad from Haskell	
Function type	**Definition**
Given a type `T` and a function `f :: A -> B`	
`fmap f :: [A] -> [B]`	`fmap f xs = [ f a | a <- xs ]`
`return :: T -> [T]`	`return x = [x]`
`join :: [[T]] -> [T]`	`join xs = concat xs`

59.4 The monad laws and their importance

Just as functors had to obey certain axioms in order to be called functors, monads have a few of their own. We'll first list them, then translate to Haskell, then see why they're important.

Given a monad $M : C \to C$ and a morphism $f : A \to B$ for $A, B \in C$,

1. $\text{join} \circ M(\text{join}) = \text{join} \circ \text{join}$
2. $\text{join} \circ M(\text{unit}) = \text{join} \circ \text{unit} = \text{id}$
3. $\text{unit} \circ f = M(f) \circ \text{unit}$
4. $\text{join} \circ M(M(f)) = M(f) \circ \text{join}$

By now, the Haskell translations should be hopefully self-explanatory:

1. `join . fmap join = join . join`
2. `join . fmap return = join . return = id`
3. `return . f = fmap f . return`
4. `join . fmap (fmap f) = fmap f . join`

(Remember that `fmap` is the part of a functor that acts on morphisms.) These laws seem a bit impenetrable at first, though. What on earth do these laws mean, and why should they be true for monads? Let's explore the laws.

[5] If you can prove that certain laws hold, which we'll explore in the next section.

59.4.1 The first law

```
join . fmap join = join . join
```

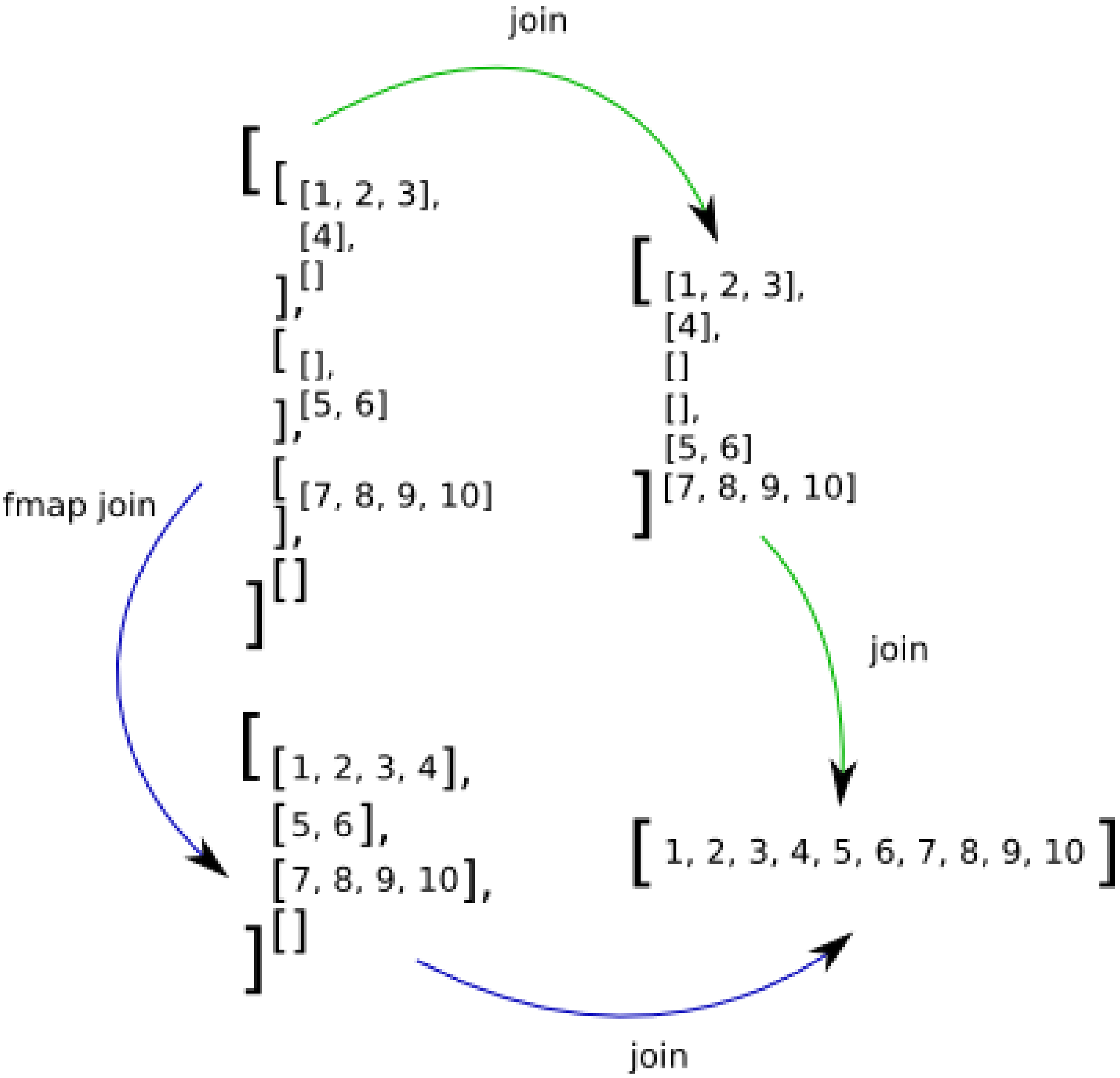


Figure 34 A demonstration of the first law for lists. Remember that `join` is `concat` and `fmap` is `map` in the list monad.

In order to understand this law, we'll first use the example of lists. The first law mentions two functions, `join . fmap join` (the left-hand side) and `join . join` (the right-hand side). What will the types of these functions be? Remembering that `join`'s type is `[[a]] -> [a]` (we're talking just about lists for now), the types are both `[a`[6]`] -> [a]` (the fact that they're the same is handy; after all, we're trying to show they're completely the same function!). So we have a list of lists of lists. The left-hand side, then, performs `fmap join` on this 3-layered list, then uses `join` on the result. `fmap` is just the familiar `map` for

[6] https://en.wikibooks.org/wiki/a

lists, so we first map across each of the list of lists inside the top-level list, concatenating them down into a list each. So afterward, we have a list of lists, which we then run through `join`. In summary, we 'enter' the top level, collapse the second and third levels down, then collapse this new level with the top level.

What about the right-hand side? We first run `join` on our list of list of lists. Although this is three layers, and you normally apply a two-layered list to `join`, this will still work, because a `[a`[7]`]` is just `[[b]]`, where `b = [a]`, so in a sense, a three-layered list is just a two layered list, but rather than the last layer being 'flat', it is composed of another list. So if we apply our list of lists (of lists) to `join`, it will flatten those outer two layers into one. As the second layer wasn't flat but instead contained a third layer, we will still end up with a list of lists, which the other `join` flattens. Summing up, the left-hand side will flatten the inner two layers into a new layer, then flatten this with the outermost layer. The right-hand side will flatten the outer two layers, then flatten this with the innermost layer. These two operations should be equivalent. It's sort of like a law of associativity for `join`.

`Maybe` is also a monad, with

```
return :: a -> Maybe a
return x = Just x

join :: Maybe (Maybe a) -> Maybe a
join Nothing         = Nothing
join (Just Nothing)  = Nothing
join (Just (Just x)) = Just x
```

So if we had a *three*-layered Maybe (i.e., it could be `Nothing`, `Just Nothing`, `Just (Just Nothing)` or `Just (Just (Just x))`), the first law says that collapsing the inner two layers first, then that with the outer layer is exactly the same as collapsing the outer layers first, then that with the innermost layer.

Exercises:
Verify that the list and Maybe monads do in fact obey this law with some examples to see precisely how the layer flattening works.

59.4.2 The second law

`join . fmap return = join . return = id`

What about the second law, then? Again, we'll start with the example of lists. Both functions mentioned in the second law are functions `[a] -> [a]`. The left-hand side expresses a function that maps over the list, turning each element `x` into its singleton list `[x]`, so that at the end we're left with a list of singleton lists. This two-layered list is flattened down into a single-layer list again using the `join`. The right hand side, however, takes the entire list `[x, y, z, ...]`, turns it into the singleton list of lists `[[x, y, z, ...]]`, then flattens the two layers down into one again. This law is less obvious to state quickly, but it essentially says that applying `return` to a monadic value, then `join`ing the result should

[7] `https://en.wikibooks.org/wiki/a`

have the same effect whether you perform the `return` from inside the top layer or from outside it.

Exercises:
Prove this second law for the Maybe monad.

59.4.3 The third and fourth laws

```
return . f = fmap f . return
```

```
join . fmap (fmap f) = fmap f . join
```

The last two laws express more self evident fact about how we expect monads to behave. The easiest way to see how they are true is to expand them to use the expanded form:

1. `\x -> return (f x) = \x -> fmap f (return x)`
2. `\x -> join (fmap (fmap f) x) = \x -> fmap f (join x)`

Exercises:
Convince yourself that these laws should hold true for any monad by exploring what they mean, in a similar style to how we explained the first and second laws.

59.4.4 Application to do-blocks

Well, we have intuitive statements about the laws that a monad must support, but why is that important? The answer becomes obvious when we consider do-blocks. Recall that a do-block is just syntactic sugar for a combination of statements involving `(>>=)` as witnessed by the usual translation:

```
do { x }                  -->  x
do { let { y = v }; x }   -->  let y = v in do { x }
do { v <- y; x }          -->  y >>= \v -> do { x }
do { y; x }               -->  y >>= \_ -> do { x }
```

Also notice that we can prove what are normally quoted as the monad laws using `return` and `(>>=)` from our above laws (the proofs are a little heavy in some cases, feel free to skip them if you want to):

1. `return x >>= f = f x`. Proof:

```
  return x >>= f
= join (fmap f (return x)) -- By the definition of (>>=)
= join (return (f x))      -- By law 3
= (join . return) (f x)
= id (f x)                 -- By law 2
= f x
```

2. `m >>= return = m`. Proof:

```
  m >>= return
= join (fmap return m)    -- By the definition of (>>=)
= (join . fmap return) m
= id m                    -- By law 2
= m
```

3. `(m >>= f) >>= g = m >>= (\x -> f x >>= g)`. Proof (recall that `fmap f . fmap g = fmap (f . g)`):

```
  (m >>= f) >>= g
= (join (fmap f m)) >>= g                          -- By the definition of
(>>=)
= join (fmap g (join (fmap f m)))                  -- By the definition of
(>>=)
= (join . fmap g) (join (fmap f m))
= (join . fmap g . join) (fmap f m)
= (join . join . fmap (fmap g)) (fmap f m)         -- By law 4
= (join . join . fmap (fmap g) . fmap f) m
= (join . join . fmap (fmap g . f)) m              -- By the distributive law
of functors
= (join . join . fmap (\x -> fmap g (f x))) m
= (join . fmap join . fmap (\x -> fmap g (f x))) m -- By law 1
= (join . fmap (join . (\x -> fmap g (f x)))) m    -- By the distributive law
of functors
= (join . fmap (\x -> join (fmap g (f x)))) m
= (join . fmap (\x -> f x >>= g)) m                -- By the definition of
(>>=)
= join (fmap (\x -> f x >>= g) m)
= m >>= (\x -> f x >>= g)                          -- By the definition of
(>>=)
```

These new monad laws, using `return` and `(>>=)`, can be translated into do-block notation.

Points-free style	Do-block style
`return x >>= f = f x`	`do { v <- return x; f v } = do { f x }`
`m >>= return = m`	`do { v <- m; return v } = do { m }`

`(m >>= f) >>= g = m >>= (\x -> f x >>= g)`	`do { y <- do { x <- m; f x };`
	`g y }`
	`=`
	`do { x <- m;`
	`y <- f x;`
	`g y }`

The monad laws are now common-sense statements about how do-blocks should function. If one of these laws were invalidated, users would become confused, as you couldn't be able to manipulate things within the do-blocks as would be expected. The monad laws are, in essence, usability guidelines.

Exercises:
In fact, the two versions of the laws we gave:

```
-- Categorical:
join . fmap join = join . join
join . fmap return = join . return = id
return . f = fmap f . return
join . fmap (fmap f) = fmap f . join

-- Functional:
m >>= return = m
return m >>= f = f m
(m >>= f) >>= g = m >>= (\x -> f x >>= g)
```

are entirely equivalent. We showed that we can recover the functional laws from the categorical ones. Go the other way; show that starting from the functional laws, the categorical laws hold. It may be useful to remember the following definitions:

```
join m = m >>= id
fmap f m = m >>= return . f
```

Thanks to Yitzchak Gale for suggesting this exercise.

59.5 Summary

We've come a long way in this chapter. We've looked at what categories are and how they apply to Haskell. We've introduced the basic concepts of category theory including functors, as well as some more advanced topics like monads, and seen how they're crucial to idiomatic Haskell. We haven't covered some of the basic category theory that wasn't needed for our aims, like natural transformations, but have instead provided an intuitive feel for the categorical grounding behind Haskell's structures.

60 The Curry-Howard isomorphism

61 fix and recursion

The `fix` function is a particularly weird-looking function when you first see it. However, it is useful for one main theoretical reason: introducing it into the (typed) lambda calculus as a primitive allows you to define recursive functions.

61.1 Introducing `fix`

Let's have the definition of `fix` before we go any further:

```
fix :: (a -> a) -> a
fix f = let x = f x in x
```

This immediately seems quite magical. Surely `fix f` will yield an infinite application stream of `f`s: `f (f (f (... )))`? The resolution to this is our good friend, *lazy evaluation*. Essentially, this sequence of applications of `f` will converge to a value if (and only if) `f` is a lazy function. Let's see some examples:

Example: `fix` examples

```
Prelude> :m Control.Monad.Fix
Prelude Control.Monad.Fix> fix (2+)
*** Exception: stack overflow
Prelude Control.Monad.Fix> fix (const "hello")
"hello"
Prelude Control.Monad.Fix> fix (1:)
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,...
```

We first import the `Control.Monad.Fix` module to bring `fix` into scope (this is also available in the `Data.Function`). Then we try some examples. Since the definition of `fix` is so simple, let's expand our examples to explain what happens:

```
  fix (2+)
= 2 + (fix (2+))
= 2 + (2 + fix (2+))
= 2 + (2 + (2 + fix (2+)))
= 2 + (2 + (2 + (2 + fix (2+))))
= ...
```

It's clear that this will never converge to any value. Let's expand the next example:

```
  fix (const "hello")
= const "hello" (fix (const "hello"))
= "hello"
```

This is quite different: we can see after one expansion of the definition of `fix` that because `const` ignores its second argument, the evaluation concludes. The evaluation for the last example is a little different, but we can proceed similarly:

```
  fix (1:)
= 1 : fix (1:)
= 1 : (1 : fix (1:))
= 1 : (1 : (1 : fix (1:)))
```

Although this similarly looks like it'll never converge to a value, keep in mind that when you type `fix (1:)` into GHCi, what it's really doing is applying `show` to that. So we should look at how `show (fix (1:))` evaluates (for simplicity, we'll pretend `show` on lists doesn't put commas between items):

```
  show (fix (1:))
= "[" ++ map show (fix (1:)) ++ "]"
= "[" ++ map show (1 : fix (1:)) ++ "]"
= "[" ++ "1" ++ map show (fix (1:)) ++ "]"
= "[" ++ "1" ++ "1" ++ map show (fix (1:)) ++ "]"
```

So although the `map show (fix (1:))` will never terminate, it does produce output: GHCi can print the beginning of the string, `"[" ++ "1" ++ "1"`, and continue to print more as `map show (fix (1:))` produces more. This is lazy evaluation at work: the printing function doesn't need to consume its entire input string before beginning to print, it does so as soon as it can start.

Lastly, iteratively calculating an approximation of a square root of a number,

```
  fix (\next guess tol val -> if abs(guess^2-val) < tol then guess else
        next ((guess + val / guess) / 2.0) tol val) 2.0 0.0001 25.0
= let f next guess tol val = if abs(guess^2-val) < tol then guess else
                             next ((guess + val / guess) / 2.0) tol val
  in fix f 2.0 0.0001 25.0
= let f ... = ...
  in f (fix f) 2.0 0.0001 25.0   -- next = fix f = f (fix f) = f next ...
= 5.000000000016778
```

Exercises:
What, if anything, will the following expressions converge to?

- `fix ("hello"++)`
- `fix (\x -> cycle (1:x))`
- `fix reverse`
- `fix id`
- `fix (\x -> take 2 $ cycle (1:x))`

61.2 fix and fixed points

A *fixed point* of a function `f` is a value `a` such that `f a == a`. For example, `0` is a fixed point of the function `(* 3)` since `0 * 3 == 0`. This is where the name of `fix` comes from: it finds the *least-defined fixed point* of a function. (We'll come to what "least defined" means in a minute.) Notice that for both of our examples above that converge, this is readily seen:

```
const "hello" "hello" -> "hello"
(1:) [1,1,..]         -> [1,1,...]
```

And since there's no number `x` such that `2+x == x`, it also makes sense that `fix (2+)` diverges.

Exercises:
For each of the functions `f` in the above exercises for which you decided that `fix f` converges, verify that `fix f` finds a fixed point.

In fact, it's obvious from the definition of `fix` that it finds a fixed point. All we need to do is write the equation for `fix` the other way around:

```
f (fix f) = fix f
```

Which is precisely the definition of a fixed point! So it seems that `fix` should always find a fixed point. But sometimes `fix` seems to fail at this, as sometimes it diverges. We can repair this property, however, if we bring in some denotational semantics[1]. Every Haskell type actually includes a special value called bottom, written ⊥. So the values with type, for example, `Int` include, in fact, ⊥ as well as `1, 2, 3` etc.. Divergent computations are denoted by a value of ⊥, i.e., we have that `fix (2+) = ⊥`.

The special value `undefined` is also denoted by this ⊥. Now we can understand how `fix` finds fixed points of functions like `(2+)`:

Example: Fixed points of (2+)

```
Prelude> (2+) undefined
*** Exception: Prelude.undefined
```

So feeding `undefined` (i.e., ⊥) to `(2+)` gives us `undefined` back. So ⊥ is a fixed point of `(2+)`!

In the case of `(2+)`, it is the only fixed point. However, there are other functions `f` with several fixed points for which `fix f` still diverges: `fix (*3)` diverges, but we remarked above that `0` is a fixed point of that function. This is where the "least-defined" clause comes

[1] Chapter 58 on page 405

in. Types in Haskell have a partial order[2] on them called *definedness*. In any type, $\perp$ is the least-defined value (hence the name "bottom"). For simple types like `Int`, the only pairs in the partial order are $\perp \leq$`1`, $\perp \leq$`2`and so on. We do not have `m ≤ n`for any non-bottom `Int`s `m`, `n`. Similar comments apply to other simple types like `Bool` and `()`. For "layered" values such as lists or `Maybe`, the picture is more complicated, and we refer to the chapter on denotational semantics[3].

So since $\perp$ is the least-defined value for all types and `fix` finds the least-defined fixed point, if `f ⊥ = ⊥`, we will have `fix f = ⊥` (and the converse is also true). If you've read the denotational semantics article, you will recognise this as the criterion for a *strict function*: `fix f` diverges if and only if `f` is strict.

61.3 Recursion

If you have already come across examples of `fix`, chances are they were examples involving `fix` and recursion. Here's a classic example:

Example: Encoding recursion with `fix`

```
Prelude> let fact n = if n == 0 then 1 else n * fact (n-1) in fact 5
120
Prelude> fix (\rec n -> if n == 0 then 1 else n * rec (n-1)) 5
120
```

Here we have used `fix` to "encode" the factorial function: note that (if we regard `fix` as a language primitive) our second definition of `fact` doesn't involve recursion at all. In a language like the typed lambda calculus that doesn't feature recursion, we can introduce `fix` in to write recursive functions in this way. Here are some more examples:

Example: More `fix` examples

```
Prelude> fix (\rec f l -> if null l then [] else f (head l) : rec f (tail l))
 (+1) [1..3]
[2,3,4]
Prelude> map (fix (\rec n -> if n == 1 || n == 2 then 1 else rec (n-1) + rec
 (n-2))) [1..10]
[1,1,2,3,5,8,13,21,34,55]
```

So how does this work? Let's first approach it from a denotational point of view with our `fact` function. For brevity let's define:

```
fact' rec n = if n == 0 then 1 else n * rec (n-1)
```

[2] `http://en.wikipedia.org/wiki/Partial_order`

[3] Chapter 58 on page 405

This is the same function as in the first example above, except that we gave a name to the anonymous function so that we're computing `fix fact' 5` now. `fix` will find a fixed point of `fact'`, i.e. the *function* `f` such that `f == fact' f`. But let's expand what this means:

```
f = fact' f
  = \n -> if n == 0 then 1 else n * f (n-1)
```

All we did was substitute `rec` for `f` in the definition of `fact'`. But this looks exactly like a *recursive* definition of a factorial function! `fix` feeds `fact'` *itself* as its first parameter in order to create a recursive function out of a higher-order function.

We can also consider things from a more operational point of view. Let's actually expand the definition of `fix fact'`:

```
  fix fact'
= fact' (fix fact')
= (\rec n -> if n == 0 then 1 else n * rec (n-1)) (fix fact')
= \n -> if n == 0 then 1 else n * fix fact' (n-1)
= \n -> if n == 0 then 1 else n * fact' (fix fact') (n-1)
= \n -> if n == 0 then 1
        else n * (\rec n' -> if n' == 0 then 1 else n' * rec (n'-1)) (fix
fact') (n-1)
= \n -> if n == 0 then 1
        else n * (if n-1 == 0 then 1 else (n-1) * fix fact' (n-2))
= \n -> if n == 0 then 1
        else n * (if n-1 == 0 then 1
                  else (n-1) * (if n-2 == 0 then 1
                                else (n-2) * fix fact' (n-3)))
= ...
```

Notice that the use of `fix` allows us to keep ”unravelling” the definition of `fact'`: every time we hit the `else` clause, we product another copy of `fact'` via the evaluation rule `fix fact' = fact' (fix fact')`, which functions as the next call in the recursion chain. Eventually we hit the `then` clause and bottom out of this chain.

Exercises:

1. Expand the other two examples we gave above in this sense. You may need a lot of paper for the Fibonacci example!
2. Write non-recursive versions of `filter` and `foldr`.

61.4 The typed lambda calculus

In this section we'll expand upon a point mentioned a few times in the previous section: how `fix` allows us to encode recursion in the typed lambda calculus. It presumes you've already met the typed lambda calculus. Recall that in the lambda calculus, there is no `let` clause or top-level bindings. Every program is a simple tree of lambda abstractions, applications and literals. Let's say we want to write a `fact` function. Assuming we have a type called `Nat` for the natural numbers, we'd start out something like the following:

```
λn:Nat. if iszero n then 1 else n * <blank> (n-1)
```

The problem is, how do we fill in the `<blank>`? We don't have a name for our function, so we can't call it recursively. The only way to bind names to terms is to use a lambda abstraction, so let's give that a go:

```
(λf:Nat→Nat. λn:Nat. if iszero n then 1 else n * f (n-1))
  (λm:Nat. if iszero m then 1 else m * <blank> (m-1))
```

This expands to:

```
λn:Nat. if iszero n then 1
        else n * (if iszero n-1 then 1 else (n-1) * <blank> (n-2))
```

We still have a `<blank>`. We could try to add one more layer in:

```
(λf:Nat→Nat. λn:Nat. if iszero n then 1 else n * f (n-1)
  ((λg:Nat→Nat. λm:Nat. if iszero n' then 1 else n' * g (m-1))
    (λp:Nat. if iszero p then 1 else p * <blank> (p-1))))

->

λn:Nat. if iszero n then 1
        else n * (if iszero n-1 then 1
                  else (n-1) * (if iszero n-2 then 1 else (n-2) * <blank>
(n-3)))
```

It's pretty clear we're never going to be able to get rid of this `<blank>`, no matter how many levels of naming we add in. Never, that is, unless we use `fix`, which, in essence, provides an object from which we can always unravel one more layer of recursion and still have what we started off:

```
fix (λf:Nat→Nat. λn:Nat. if iszero n then 1 else n * f (n-1))
```

This is a perfect factorial function in the typed lambda calculus plus `fix`.

`fix` is actually slightly more interesting than that in the context of the typed lambda calculus: if we introduce it into the language, then every type becomes inhabited, because given some concrete type `T`, the following expression has type `T`:

```
fix (λx:T. x)
```

This, in Haskell-speak, is `fix id`, which is denotationally ⊥. So we see that as soon as we introduce `fix` to the typed lambda calculus, the property that every well-typed term reduces to a value is lost.

61.5 Fix as a data type

It is also possible to make a fix data type in Haskell.

There are three ways of defining it.

```
newtype Fix f = Fix (f (Fix f))
```

or using the RankNTypes extension

```
newtype Mu f=Mu (forall a.(f a->a)->a)
data Nu f=forall a.Nu a (a->f a)
```

Mu and Nu help generalize folds, unfolds and refolds.

```
fold :: (f a -> a) -> Mu f -> a
fold g (Mu f)=f g
unfold :: (a -> f a) -> a -> Nu f
unfold f x=Nu x f
refold :: (a -> f a) -> (g a-> a) -> Mu f -> Nu g
refold f g=unfold g . fold f
```

Mu and Nu are restricted versions of Fix. Mu is used for making inductive noninfinite data and Nu is used for making coinductive infinite data. Eg)

```
newpoint Stream a=Stream (Nu ((,) a)) -- forsome b. (b,b->(a,b))
newpoint Void a=Void (Mu ((,) a)) -- forall b.((a,b)->b)->b
```

Unlike the fix point function the fix point types do not lead to bottom. In the following code Bot is perfectly defined. It is equivalent to the unit type ().

```
newtype Id a=Id a
newtype Bot=Bot (Fix Id) -- equals          newtype Bot=Bot Bot
-- There is only one allowable term. Bot $ Bot $ Bot $ Bot ..,
```

The Fix data type cannot model all forms of recursion. Take for instance this nonregular data type.

```
data Node a=Two a a|Three a a a
data FingerTree a=U a|Up (FingerTree (Node a))
```

It is not easy to implement this using Fix.

62 Haskell Performance

63 Introduction

1. REDIRECT Haskell/Performance introduction[1]

[1] Chapter 63 on page 459

64 Step by Step Examples

1. REDIRECT Haskell/Performance examples[1]

[1] Chapter 64 on page 461

65 Graph reduction

65.1 Notes and TODOs

- *TODO: Pour lazy evaluation explanation from ../Laziness/*[1] *into this mold.*
- *TODO: better section names.*
- *TODO: ponder the graphical representation of graphs.*
  - *No grapical representation, do it with* `let .. in`. *Pro: Reduction are easiest to perform in that way anyway. Cons: no graphic.*
  - *ASCII art / line art similar to the one in Bird&Wadler? Pro: displays only the relevant parts truly as graph, easy to perform on paper. Cons: Ugly, no large graphs with that.*
  - *Full blown graphs with @-nodes? Pro: look graphy. Cons: nobody needs to know @-nodes in order to understand graph reduction. Can be explained in the implementation section.*
  - *Graphs without @-nodes. Pro: easy to understand. Cons: what about currying?*
- *! Keep this chapter short. The sooner the reader knows how to evaluate Haskell programs by hand, the better.*
- *First sections closely follow Bird&Wadler*

65.2 Introduction

Programming is not only about writing correct programs, answered by denotational semantics, but also about writing fast ones that require little memory. For that, we need to know how they're executed on a machine, commonly given by operational semantics. This chapter explains how Haskell programs are commonly executed on a real computer and thus serves as foundation for analyzing time and space usage. Note that the Haskell standard deliberately does *not* give operational semantics, implementations are free to choose their own. But so far, every implementation of Haskell more or less closely follows the execution model of *lazy evaluation.*

In the following, we will detail lazy evaluation and subsequently use this execution model to explain and exemplify the reasoning about time and memory complexity of Haskell programs.

[1] Chapter 66 on page 475

65.3 Evaluating Expressions by Lazy Evaluation

65.3.1 Reductions

Executing a functional program, i.e. evaluating an expression, means to repeatedly apply function definitions until all function applications have been expanded. Take for example the expression `pythagoras 3 4` together with the definitions

```
      square x = x * x
pythagoras x y = square x + square y
```

One possible sequence of such **reduction**s is

```
pythagoras 3 4
 ⇒ square 3 + square 4    (pythagoras)
 ⇒    (3*3) + square 4    (square)
 ⇒        9 + square 4    (*)
 ⇒        9 + (4*4)       (square)
 ⇒        9 + 16          (*)
 ⇒          25
```

Every reduction replaces a subexpression, called **reducible expression** or **redex** for short, with an equivalent one, either by appealing to a function definition like for `square` or by using a built-in function like `(+)`. An expression without redexes is said to be in **normal form**. Of course, execution stops once reaching a normal form which thus is the result of the computation.

Clearly, the fewer reductions that have to be performed, the faster the program runs. We cannot expect each reduction step to take the same amount of time because its implementation on real hardware looks very different, but in terms of asymptotic complexity, this number of reductions is an accurate measure.

65.3.2 Reduction Strategies

There are many possible reduction sequences and the number of reductions may depend on the order in which reductions are performed. Take for example the expression `fst (square 3, square 4)`. One systematic possibility is to evaluate all function arguments before applying the function definition

```
fst (square 3, square 4)
 ⇒ fst (3*3, square 4)    (square)
 ⇒ fst ( 9 , square 4)    (*)
 ⇒ fst ( 9 , 4*4)         (square)
 ⇒ fst ( 9 , 16 )         (*)
 ⇒ 9                      (fst)
```

This is called an **innermost reduction** strategy and an **innermost redex** is a redex that has no other redex as subexpression inside.

Another systematic possibility is to apply all function definitions first and only then evaluate arguments:

```
fst (square 3, square 4)
 ⇒  square 3            (fst)
 ⇒  3*3                 (square)
 ⇒  9                   (*)
```

which is named **outermost reduction** and always reduces **outermost redex**es that are not inside another redex. Here, the outermost reduction uses fewer reduction steps than the innermost reduction. Why? Because the function `fst` doesn't need the second component of the pair and the reduction of `square 4` was superfluous.

65.3.3 Termination

For some expressions like

```
loop = 1 + loop
```

no reduction sequence may terminate and program execution enters a neverending loop, those expressions do not have a normal form. But there are also expressions where some reduction sequences terminate and some do not, an example being

```
fst (42, loop)
 ⇒  42                  (fst)

fst (42, loop)
 ⇒  fst (42,1+loop)     (loop)
 ⇒  fst (42,1+(1+loop)) (loop)
 ⇒  ...
```

The first reduction sequence is outermost reduction and the second is innermost reduction which tries in vain to evaluate the `loop` even though it is ignored by `fst` anyway. The ability to evaluate function arguments only when needed is what makes outermost optimal when it comes to termination:

Theorem (Church Rosser II)

If there is one terminating reduction, then outermost reduction will terminate, too.

65.3.4 Graph Reduction (Reduction + Sharing)

Despite the ability to discard arguments, outermost reduction doesn't always take fewer reduction steps than innermost reduction:

```
square (1+2)
 ⇒  (1+2)*(1+2)         (square)
 ⇒  (1+2)*3             (+)
 ⇒      3*3             (+)
 ⇒       9              (*)
```

Here, the argument (1+2) is duplicated and subsequently reduced twice. But because it is one and the same argument, the solution is to share the reduction (1+2) ⇒ 3 with all other incarnations of this argument. This can be achieved by representing expressions as *graphs*. For example,

```
 __________
|   |    ↓
◊ * ◊     (1+2)
```

represents the expression (1+2)*(1+2). Now, the **outermost graph reduction** of square (1+2) proceeds as follows

```
square (1+2)
 ⇒  __________           (square)
   |   |    ↓
   ◊ * ◊     (1+2)
 ⇒  __________           (+)
   |   |    ↓
   ◊ * ◊      3

 ⇒ 9                     (*)
```

and the work has been shared. In other words, outermost graph reduction now reduces every argument at most once. For this reason, it always takes fewer reduction steps than the innermost reduction, a fact we will prove when reasoning about time[2].

Sharing of expressions is also introduced with let and where constructs. For instance, consider Heron's formula[3] for the area of a triangle with sides a,b and c:

```
area a b c = let s = (a+b+c)/2 in
     sqrt (s*(s-a)*(s-b)*(s-c))
```

Instantiating this to an equilateral triangle will reduce as

```
area 1 1 1
 ⇒        _____________________            (area)
        |  |    |     |      ↓
   sqrt (◊*(◊-a)*(◊-b)*(◊-c))  ((1+1+1)/2)
 ⇒        _____________________            (+),(+),(/)
        |  |    |     |      ↓
   sqrt (◊*(◊-a)*(◊-b)*(◊-c))  1.5
 ⇒
   ...
 ⇒
   0.433012702
```

[2] Chapter 65.5 on page 472

[3] https://en.wikipedia.org/wiki/Heron%27s%20formula

which is $\sqrt{3}/4$. Put differently, `let`-bindings simply give names to nodes in the graph. In fact, one can dispense entirely with a graphical notation and solely rely on `let` to mark sharing and express a graph structure.[4]

Any implementation of Haskell is in some form based on outermost graph reduction which thus provides a good model for reasoning about the asympotic complexity of time and memory allocation. The number of reduction steps to reach normal form corresponds to the execution time and the size of the terms in the graph corresponds to the memory used.

Exercises:

1. Reduce `square (square 3)` to normal form with innermost and outermost graph reduction.
2. Consider the fast exponentiation algorithm

   ```
   power x 0 = 1
   power x n = x' * x' * (if n `mod` 2 == 0 then 1 else x)
     where x' = power x (n `div` 2)
   ```

   that takes `x` to the power of `n`. Reduce `power 2 5` with innermost and outermost graph reduction. How many reductions are performed? What is the asymptotic time complexity for the general `power 2 n`? What happens to the algorithm if we use ”graphless” outermost reduction?

65.3.5 Pattern Matching

So far, our description of outermost graph reduction is still underspecified when it comes to pattern matching and data constructors. Explaining these points will enable the reader to trace most cases of the reduction strategy that is commonly the base for implementing non-strict functional languages like Haskell. It is called **call-by-need** or **lazy evaluation** in allusion to the fact that it ”lazily” postpones the reduction of function arguments to the last possible moment. Of course, the remaining details are covered in subsequent chapters.

To see how pattern matching needs specification, consider for example the boolean disjunction

```
or True  y = True
or False y = y
```

and the expression

[4] John Maraist, Martin Odersky, and Philip Wadler . The call-by-need lambda calculus The call-by-need lambda calculus ^{homepages.inf.ed.ac.uk/wadler/topics/call-{}by-{}need.html# need-{}journal} . *Journal of Functional Programming* , **8** : 257-317 May 1998 http://homepages.inf.ed.ac.uk/wadler/topics/call-by-need.html#need-journal

```
or (1==1) loop
```

with a non-terminating `loop = not loop`. The following reduction sequence

```
or (1==1) loop
 ⇒ or (1==1) (not loop)        (loop)
 ⇒ or (1==1) (not (not loop))  (loop)
 ⇒ ...
```

only reduces outermost redexes and therefore is an outermost reduction. But

```
or (1==1) loop
 ⇒ or True   loop              (or)
 ⇒ True
```

makes much more sense. Of course, we just want to apply the definition of `or` and are only reducing arguments to decide which equation to choose. This intention is captured by the following rules for pattern matching in Haskell:

- Left hand sides are matched from top to bottom
- When matching a left hand side, arguments are matched from left to right
- Evaluate arguments only as much as needed to decide whether they match or not.

Thus, for our example `or (1==1) loop`, we have to reduce the first argument to either `True` or `False`, then evaluate the second to match a variable `y` pattern and then expand the matching function definition. As the match against a variable always succeeds, the second argument will not be reduced at all. It is the second reduction section above that reproduces this behavior.

With these preparations, the reader should now be able to evaluate most Haskell expressions. Here are some random encounters to test this ability:

Exercises:
Reduce the following expressions with lazy evaluation to normal form. Assume the standard function definitions from the Prelude.

- `length [42,42+1,42-1]`
- `head (map (2*) [1,2,3])`
- `head $ [1,2,3] ++ (let loop = tail loop in loop)`
- `zip [1..3] (iterate (+1) 0)`
- `head $ concatMap (\x -> [x,x+1]) [1,2,3]`
- `take (42-6*7) $ map square [2718..3146]`

65.3.6 Higher Order Functions

The remaining point to clarify is the reduction of higher order functions and currying. For instance, consider the definitions

```
id x = x
a = id (+1) 41
```

```
twice f = f . f
b = twice (+1) (13*3)
```

where both `id` and `twice` are only defined with one argument. The solution is to see multiple arguments as subsequent applications to one argument, this is called **currying**

```
a = (id    (+1)) 41
b = (twice (+1)) (13*3)
```

To reduce an arbitrary application *`expression`*$_1$ *`expression`*$_2$, call-by-need first reduce *expression*$_1$ until this becomes a function whose definition can be unfolded with the argument *`expression`*$_2$. Hence, the reduction sequences are

```
a
 ⇒ (id (+1)) 41         (a)
 ⇒ (+1) 41              (id)
 ⇒ 42                   (+)

b
 ⇒ (twice (+1)) (13*3)   (b)
 ⇒ ((+1).(+1) ) (13*3)   (twice)
 ⇒ (+1) ((+1) (13*3))    (.)
 ⇒ (+1) ((+1)  39)       (*)
 ⇒ (+1) 40               (+)
 ⇒ 41                    (+)
```

Admittedly, the description is a bit vague and the next section will detail a way to state it clearly.

While it may seem that pattern matching is the workhorse of time intensive computations and higher order functions are only for capturing the essence of an algorithm, functions are indeed useful as data structures. One example are difference lists (`[a] -> [a]`) that permit concatenation in $O(1)$ time, another is the representation of a stream by a fold. In fact, all data structures are represented as functions in the pure lambda calculus, the root of all functional programming languages.

Exercises! Or not? Diff-Lists Best done with `foldl (++)` but this requires knowledge of the fold example. Oh, where do we introduce the foldl VS. foldr example at all? Hm, Bird&Wadler sneak in an extra section "Meet again with fold" for the (++) example at the end of "Controlling reduction order and space requirements" :-/ The complexity of (++) is explained when arguing about `reverse`.

65.3.7 Weak Head Normal Form

To formulate precisely how lazy evaluation chooses its reduction sequence, it is best to abandon equational function definitions and replace them with an expression-oriented approach. In other words, our goal is to translate function definitions like `f (x:xs) = ...` into the form `f = `*`expression`*. This can be done with two primitives, namely case-expressions and lambda abstractions.

In their primitive form, case-expressions only allow the discrimination of the outermost constructor. For instance, the primitive case-expression for lists has the form

```
case expression of
 []   -> ...
 x:xs -> ...
```

Lambda abstractions are functions of one parameter, so that the following two definitions are equivalent

```
f x = expression
f   = \x -> expression
```

Here is a translation of the definition of `zip`

```
zip :: [a] -> [a] -> [(a,a)]
zip []      ys      = []
zip xs      []      = []
zip (x:xs') (y:ys') = (x,y):zip xs' ys'
```

to case-expressions and lambda-abstractions:

```
zip = \xs -> \ys ->
   case xs of
      []    -> []
      x:xs' ->
         case ys of
            []    -> []
            y:ys' -> (x,y):zip xs' ys'
```

Assuming that all definitions have been translated to those primitives, every redex now has the form of either

- a function application `(\`*`variable`*`->`*`expression`*$_1$`)` *`expression`*$_2$
- or a case-expression `case `*`expression`*` of { ... }`

lazy evaluation.

Weak Head Normal Form

An expression is in weak head normal form, iff it is either

- a constructor (possibly applied to arguments) like `True`, `Just (square 42)` or `(:) 1`
- a built-in function applied to too few arguments (perhaps none) like `(+) 2` or `sqrt`.
- or a lambda abstraction `\x -> `*`expression`*.

functions types cannot be pattern matched anyway, but the devious seq can evaluate them to WHNF nonetheless. "weak" = no reduction under lambdas. "head" = first the function application, then the arguments.

65.3.8 Strict and Non-strict Functions

A non-strict function doesn't need its argument. A strict function needs its argument in WHNF, as long as we do not distinguish between different forms of non-termination (`f x = loop`* doesn't need its argument, for example).*

65.4 Controlling Space

”Space” here may be better visualized as traversal of a graph. Either a data structure, or an induced dependencies graph. For instance : Fibonacci(N) depends on : Nothing if N = 0 or N = 1 ; Fibonacci(N-1) and Fibonacci(N-2) else. As Fibonacci(N-1) depends on Fibonacci(N-2), the induced graph is not a tree. Therefore, there is a correspondence between implementation technique and data structure traversal :

Corresponding Implementation technique	**Data Structure Traversal**
Memoization	Depth First Search (keep every intermediary result in memory)
Parallel evaluation	Breadth First Search (keep every intermediary result in memory, too)
Sharing	Directed acyclic graph traversal (Maintain only a ”frontier” in memory.)
Usual recursion	Tree traversal (Fill a stack)
Tail recursion	List traversal / Greedy Search (Constant space)

The classical :

```
fibo 0 = 1
fibo 1 = 1
fibo n = fibo (n-1) + fibo (n-2)
```

Is a tree traversal applied to a directed acyclic graph for the worse. The optimized version :

```
fibo n =
 let f a b m =
    if m = 0 then a
    if m = 1 then b
    f b (a+b) (m-1)
 in f 1 1 n
```

Uses a DAG traversal. Luckily, the frontier size is constant, so it's a tail recursive algorithm.

NOTE: The chapter ../Strictness[5] *is intended to elaborate on the stuff here.*

[5] Chapter 67 on page 487

NOTE: The notion of strict function is to be introduced before this section.

Now’s the time for the space-eating fold example:

```
foldl (+) 0 [1..10]
```

Introduce `seq` and `$!` that can force an expression to WHNF. => `foldl'`.

Tricky space leak example:

```
(\xs -> head xs + last xs) [1..n]
(\xs -> last xs + head xs) [1..n]
```

The first version runs on O(1) space. The second in O(n).

65.4.1 Sharing and CSE

NOTE: overlaps with section about time. Hm, make an extra memoization section?

How to share

```
foo x y = -- s is not shared
foo x = \y -> s + y
  where s = expensive x -- s is shared
```

”Lambda-lifting”, ”Full laziness”. The compiler should not do full laziness.

A classic and important example for the trade between space and time:

```
sublists []      = [6]
sublists (x:xs)  = sublists xs ++ map (x:) (sublists xs)
sublists' (x:xs) = let ys = sublists' xs in ys ++ map (x:) ys
```

That’s why the compiler should not do common subexpression elimination as optimization. (Does GHC?).

65.4.2 Tail recursion

NOTE: Does this belong to the space section? I think so, it’s about stack space.

Tail recursion in Haskell looks different.

65.5 Reasoning about Time

Note: introducing strictness before the upper time bound saves some hassle with explanation?

65.5.1 Lazy eval < Eager eval

When reasoning about execution time, naively performing graph reduction by hand to get a clue on what's going on is most often infeasible. In fact, the order of evaluation taken by lazy evaluation is difficult to predict by humans, it is much easier to trace the path of eager evaluation where arguments are reduced to normal form before being supplied to a function. But knowing that lazy evaluation always performs fewer reduction steps than eager evaluation (present the proof!), we can easily get an upper bound for the number of reductions by pretending that our function is evaluated eagerly.

Example:

```
or = foldr (||) False
isPrime n = not $ or $ map (\k -> n `mod` k == 0) [2..n-1]
```

=> eager evaluation always takes n steps, lazy won't take more than that. But it will actually take fewer.

65.5.2 Throwing away arguments

Time bound exact for functions that examine their argument to normal form anyway. The property that a function needs its argument can concisely be captured by denotational semantics:

```
f ⊥ = ⊥
```

Argument in WHNF only, though. Operationally: non-termination -> non-termination. (this is an approximation only, though because f anything = ⊥ doesn't "need" its argument). Non-strict functions don't need their argument and eager time bound is not sharp. But the information whether a function is strict or not can already be used to great benefit in the analysis.

```
isPrime n = not $ or $ (n `mod` 2 == 0) : (n `mod` 3 == 0) : ...
```

It's enough to know `or True ⊥ = True`.

Other examples:

- `foldr (:) []` vs. `foldl (flip (:)) []` with ⊥.
- Can `head . mergesort` be analyzed only with ⊥? In any case, this example is too involed and belongs to ../Laziness[7].

[7] Chapter 66 on page 475

65.5.3 Persistence & Amortisation

NOTE: this section is better left to a data structures chapter because the subsections above cover most of the cases a programmer not focussing on data structures / amortization will encounter.

Persistence = no updates in place, older versions are still there. Amortisation = distribute unequal running times across a sequence of operations. Both don't go well together in a strict setting. Lazy evaluation can reconcile them. Debit invariants. Example: incrementing numbers in binary representation.

65.6 Implementation of Graph reduction

Small talk about G-machines and such. Main definition:

closure = thunk = code/data pair on the heap. What do they do? Consider $(\lambda x.\lambda y.x+y)2$. This is a function that returns a function, namely $\lambda y.2+y$ in this case. But when you want to compile code, it's prohibitive to actually perform the substitution in memory and replace all occurrences of x by 2. So, you return a closure that consists of the function code $\lambda y.x+y$ and an environment $\{x=2\}$ that assigns values to the free variables appearing in there.

GHC (?, most Haskell implementations?) avoid free variables completely and use supercombinators. In other words, they're supplied as extra-parameters and the observation that lambda-expressions with too few parameters don't need to be reduced since their WHNF is not very different.

Note that these terms are technical terms for implementation stuff, lazy evaluation happily lives without them. Don't use them in any of the sections above.

65.7 References

- Introduction to Functional Programming using Haskell . Prentice Hall , , 1998
- The Implementation of Functional Programming Languages . Prentice Hall , , 1987

66 Laziness

Quote:
Hard work pays off later. Laziness pays off now! – Steven Wright

66.1 Introduction

By now, you are aware that Haskell uses lazy evaluation in that nothing is evaluated until necessary. But what exactly does ”until necessary” mean? In this chapter, we will see how lazy evaluation works (how little black magic there is), what exactly it means for functional programming, and how to make the best use of it.

First, let’s consider the reasons and implications of lazy evaluation. At first glance, we might think that lazy evaluation makes programs more efficient. After all, what can be more efficient than not doing anything? In practice, however, laziness often introduces an overhead that leads programmers to hunt for places where they can make their code more strict. The real benefit of laziness is in making the right things **efficient enough**. *Lazy evaluation allows us to write more simple, elegant code than we could in a strict environment.*

66.1.1 Nonstrictness versus Laziness

There is a slight difference between *laziness* and *nonstrictness*. **Nonstrict semantics** refers to a given property of Haskell programs that you can rely on: nothing will be evaluated until it is needed. **Lazy evaluation** is how you implement nonstrictness using a device called **thunks** which we explain in the next section. However, these two concepts are so closely linked that it helps to explain them both together. Knowledge of thunks helps in understanding nonstrictness, and the semantics of nonstrictness explains why we use lazy evaluation in the first place. So, we’ll introduce the concepts simultaneously and make no particular effort to keep them from intertwining (with the exception of getting the terminology right).

66.2 Thunks and Weak head normal form

You need to understand two principles to see how programs execute in Haskell. First, we have the property of nonstrictness: we evaluate as little as possible and delay evaluation as long as possible. Second, Haskell values are highly layered; and ’evaluating’ a Haskell value could mean evaluating down to any one of these layers. Let’s walk through a few examples using a pair.

```
let (x, y) = (length [1..5], reverse "olleh") in ...
```

Assume that in the 'in' part, we use x and y somewhere — otherwise, we wouldn't need to evaluate the let-binding at all! The right-hand side could have been `undefined`, and it would still work if the 'in' part doesn't mention x or y. This assumption will remain for all the examples in this section.

What do we know about x? We can calculate that x must be 5 and y is "hello", but remember the first principle: we don't evaluate the calls to `length` and `reverse` until we're forced to. So, we say that x and y are both **thunks**: that is, they are *unevaluated values* with a *recipe* that explains how to evaluate them. For example, for x this recipe says 'Evaluate `length [1..5]`'. However, we are actually doing some pattern matching on the left hand side. What would happen if we removed that?

```
let z = (length [1..5], reverse "olleh") in ...
```

Although it's still pretty obvious to us that z is a pair, the compiler sees that we're not trying to deconstruct the value on the right-hand side of the '=' sign at all, so it doesn't really care what's there. It lets z be a thunk on its own. Later on, when we try to use z, we'll probably need one or both of the components, so we'll have to evaluate z, but for now, it can be a thunk.

Above, we said Haskell values were layered. We can see that at work if we pattern match on z:

```
let z      = (length [1..5], reverse "olleh")
   (n, s) = z
in ...
```

After the first line has been executed, z is simply a thunk. We know nothing about the sort of value it is because we haven't been asked to find out yet. In the second line, however, we pattern match on z using a pair pattern. The compiler thinks 'I better make sure that pattern does indeed match z, and in order to do that, I need to make sure z is a pair.' Be careful, though — we're not yet doing anything with the component parts (the calls to `length` and `reverse`), so they can remain unevaluated. In other words, z, which was just a thunk, gets evaluated to something like `(*thunk*, *thunk*)`, and n and s become thunks which, when evaluated, will be the component parts of the original z.

Let's try a slightly more complicated pattern match:

```
let z      = (length [1..5], reverse "olleh")
   (n, s) = z
   'h':ss = s
in ...
```

The pattern match on the second component of z causes some evaluation. The compiler wishes to check that the `'h':ss` pattern matches the second component of the pair. So, it:

1. Evaluates the top level of s to ensure it's a cons cell: `s = *thunk* : *thunk*`. (If s had been an empty list we would encounter a pattern match failure error at this point.)
2. Evaluates the first thunk it just revealed to make sure it's `'h':ss = 'h' : *thunk*`

- The rest of the list stays unevaluated, and ss becomes a thunk which, when evaluated, will be the rest of this list.

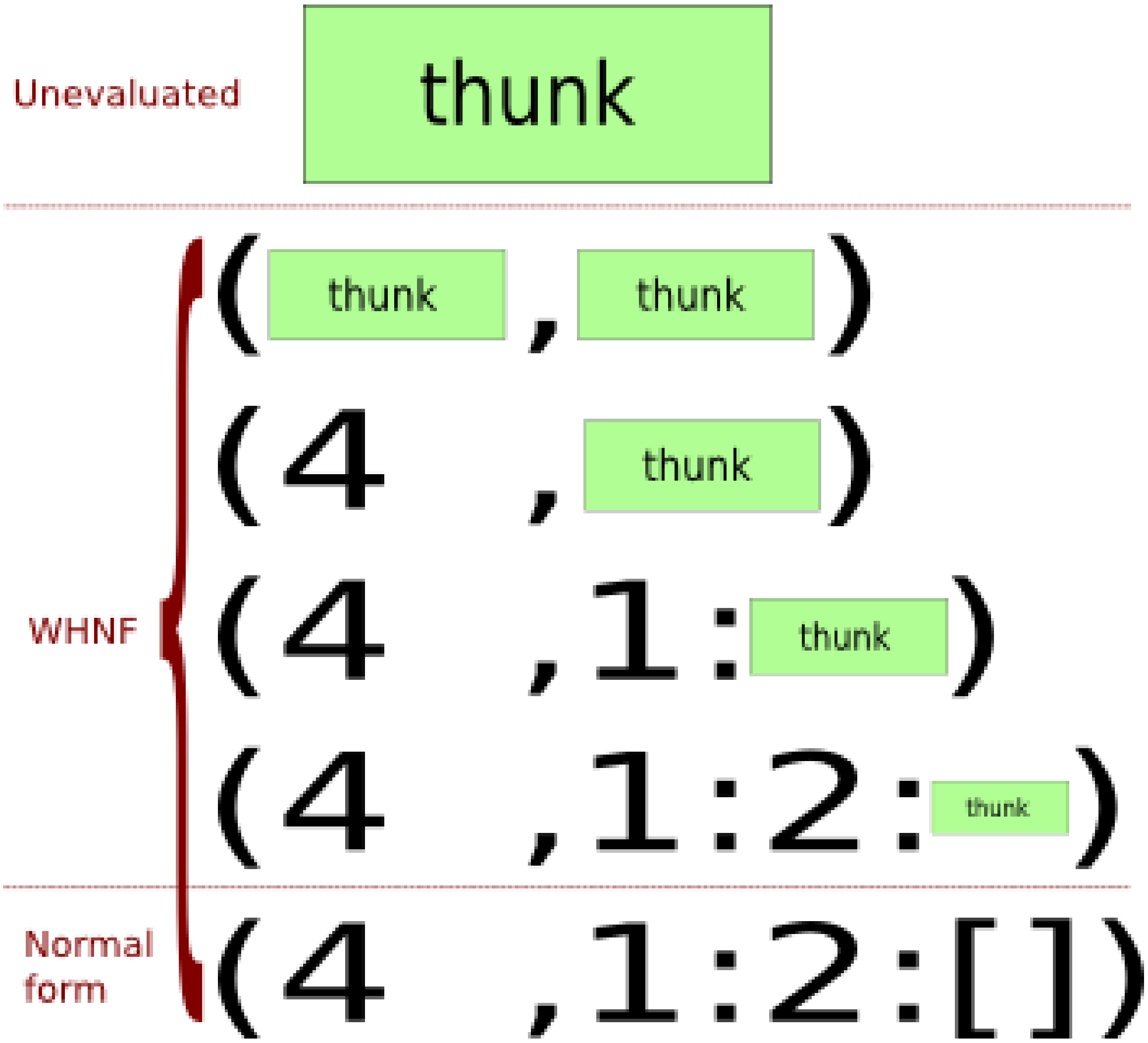


Figure 35 Evaluating the value `(4, [1, 2])` step by step. The first stage is completely unevaluated; all subsequent forms are in WHNF, and the last one is also in normal form.

We can 'partially evaluate' (most) Haskell values. Also, there is some sense of the minimum amount of evaluation we can do. For example, if we have a pair thunk, then the minimum amount of evaluation takes us to the pair constructor with two unevaluated components: `(*thunk*, *thunk*)`. If we have a list, the minimum amount of evaluation takes us either to a cons cell `*thunk* : *thunk*` or an empty list `[]`. Note that in the empty list case, no more evaluation can be performed on the value; it is said to be in **normal form**. If we are at any of the intermediate steps so that we've performed at least some evaluation on a value, it is in **weak head normal form** (WHNF). (There is also a 'head normal form', but it's not used in Haskell.) *Fully* evaluating something in WHNF reduces it to something in normal form; if at some point we needed to, say, print `z` out to the user, we'd need to fully evaluate it, including those calls to `length` and `reverse`, to `(5, "hello")`, where it is in normal form. Performing any degree of evaluation on a value is sometimes called **forcing** that value.

Note that for some values there is only one result. For example, you can't partially evaluate an integer. It's either a thunk or it's in normal form. Furthermore, if we have a constructor with strict components (annotated with an exclamation mark, as with `data MaybeS a = NothingS | JustS !a`), these components become evaluated as soon as we evaluate the level above. I.e. we can never have `JustS *thunk*` — as soon as we get to this level, the strictness annotation on the component of `JustS` forces us to evaluate the component part.

In this section we've explored the basics of laziness. We've seen that nothing gets evaluated until it is needed (in fact, the *only* place that Haskell values get evaluated is in pattern matches and inside certain primitive IO functions) . This principle even applies to evaluating values — we do the minimum amount of work on a value that we need to compute our result.

66.3 Lazy and strict functions

Functions can be lazy or strict 'in an argument'. Most functions need to do something with their arguments, and this will involve evaluating these arguments to different levels. For example, `length` needs to evaluate only the cons cells in the argument you give it, not the contents of those cons cells. `length *thunk*` might evaluate to something like `length (*thunk* : *thunk* : *thunk* : [])`, which in turn evaluates to `3`. Others need to evaluate their arguments fully, as in `(length . show)`. If you had `length $ show *thunk*`, there's no way you can do anything other than evaluate that thunk to normal form.

So, some functions evaluate their arguments more fully than others. Given two functions of one parameter, `f` and `g`, we say `f` is stricter than `g` if `f x` evaluates `x` to a deeper level than `g x`. Often, we only care about WHNF, so a function that evaluates its argument to at least WHNF is called *strict* and one that performs no evaluation is *lazy*. What about functions of more than one parameter? Well, we can talk about functions being strict in one parameter, but lazy in another. For example, given a function like the following:

```
f x y = length $ show x
```

Clearly we need to perform no evaluation on `y`, but we need to evaluate `x` fully to normal form, so `f` is strict in its first parameter but lazy in its second.

Exercises:

1. Why must we fully evaluate x to normal form in f x y = show x?
2. Which is the stricter function?

```
f x = length [head x]
g x = length (tail x)
```

In the original discussion about Folds[1], we discussed memory problems with `foldl` that are solved by the strictly-evaluated `foldl'`. Essentially, `foldr (:) []` and `foldl (flip (:)) []` both evaluate their arguments to the same level of strictness, but `foldr` can start producing values straight away, whereas `foldl` needs to evaluate cons cells all the way to the end before it starts producing any output. So, there are times when strictness can be valuable.

66.3.1 Black-box strictness analysis

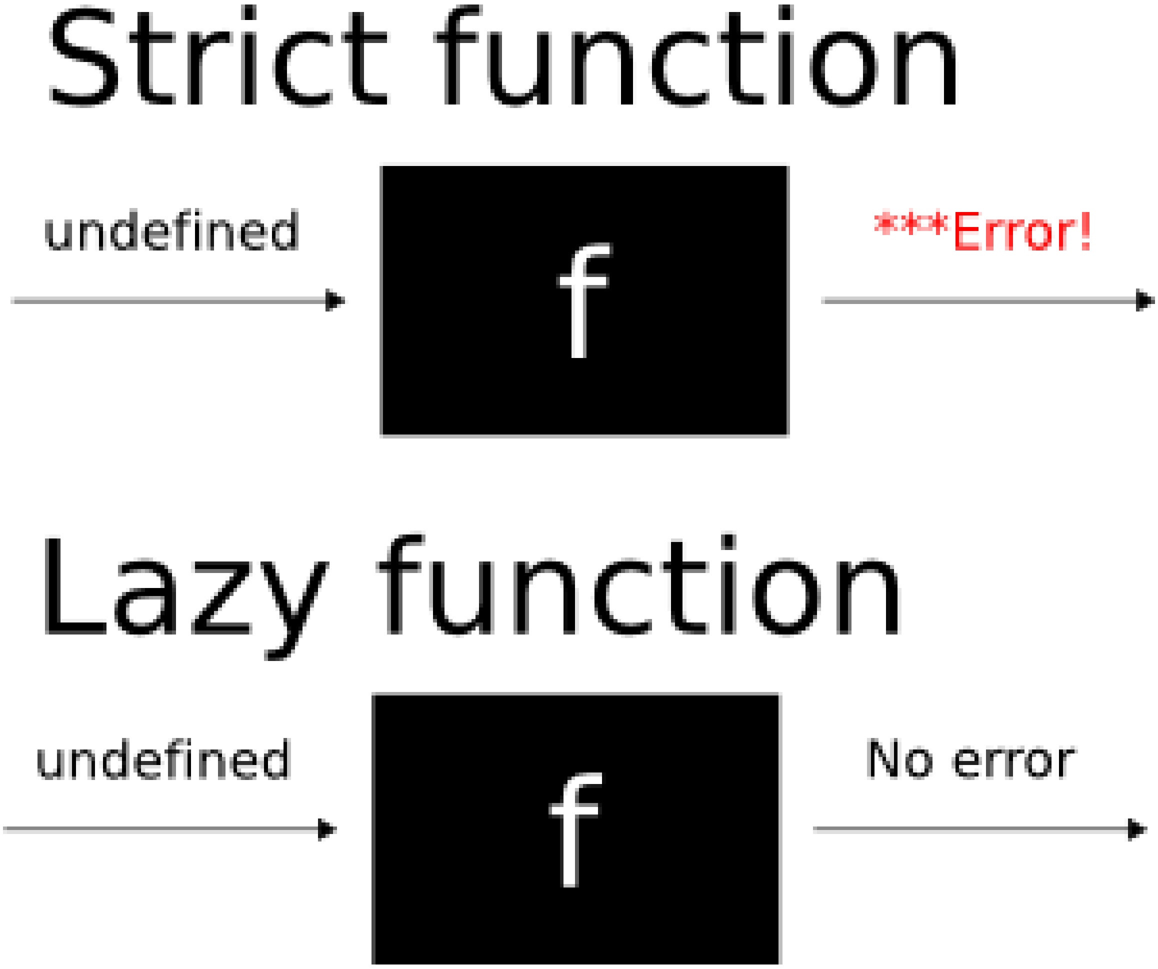


Figure 36 If `f` returns an error when passed undefined, it must be strict. Otherwise, it's lazy.

Imagine we're given some function `f` which takes a single parameter. We're not allowed to look at its source code, but we want to know whether `f` is strict or not. How might we do this? Probably the easiest way is to use the standard Prelude value `undefined`. Forcing `undefined` to any level of evaluation will halt our program and print an error, so all of these print errors:

[1] https://en.wikibooks.org/wiki/Haskell%2FList%20processing%23foldl

```
let (x, y) = undefined in x
length undefined
head undefined
JustS undefined -- Using MaybeS as defined in the last section
```

So if a function is strict, passing it `undefined` will result in an error. Were the function lazy, passing it undefined would print no error and we can carry on as normal. For example, none of the following produce errors:

```
let (x, y) = (4, undefined) in x
length [undefined, undefined, undefined]
head (4 : undefined)
Just undefined
```

So we can say that `f` is a strict function if, and only if, `f undefined` results in an error being printed and the halting of our program.

66.3.2 In the context of nonstrict semantics

What we've presented so far makes sense until you start to think about functions like `id`. Is `id` strict? Our gut reaction is probably to say ”No! It doesn't evaluate its argument, therefore it's lazy”. However, let's apply our black-box strictness analysis from the last section to `id`. Clearly, `id undefined` is going to print an error and halt our program, so shouldn't we say that `id` is strict? The reason for this mixup is that Haskell's nonstrict semantics makes the whole issue a bit murkier.

Nothing gets evaluated if it doesn't need to be, according to nonstrictness. In the following code, will `length undefined` be evaluated?

```
[4, 10, length undefined, 12]
```

If you type this into GHCi, it seems strict — you'll get an error. However, our question was something of a trick. It doesn't make sense to state whether a value gets evaluated unless we're doing something to this value. Think about it: if we type in `head [1, 2, 3]` into GHCi, the only reason we have to do any evaluation whatsoever is because GHCi has to print us out the result. Typing `[4, 10, length undefined, 12]` again requires GHCi to print that list back to us, so it must evaluate it to normal form. In your average Haskell program, nothing at all will be evaluated until we come to perform the IO in `main`. So it makes no sense to say whether something is evaluated or not unless we know what it's being passed to, one level up. One lesson here is: don't blindly trust GHCi because *everything in GHCi is filtered through IO!*

So when we say ”Does `f x` force `x`?” what we really mean is ”Given that we're forcing `f x`, does `x` get forced as a result?”. Now we can turn our attention back to `id`. If we force `id x` to normal form, then `x` will be forced to normal form, so we conclude that `id` is strict. `id` itself doesn't evaluate its argument, it just hands it on to the caller who will. One way to see this is in the following code:

```
-- We evaluate the right-hand of the let-binding to WHNF by pattern-matching
-- against it.
let (x, y) = undefined in x -- Error, because we force undefined.
let (x, y) = id undefined in x -- Error, because we force undefined.
```

`id` doesn't "stop" the forcing, so it is strict. Contrast this to a clearly lazy function, `const (3, 4)`:

```
let (x, y) = undefined in x -- Error, because we force undefined.
let (x, y) = const (3, 4) undefined -- No error, because const (3, 4) is lazy.
```

66.3.3 The denotational view on things

If you're familiar with denotational semantics (perhaps you've read the wikibook chapter[2]?), then the strictness of a function can be summed up very succinctly:

$$f \perp = \perp \Leftrightarrow f \text{ is strict}$$

Assuming that, we can say that everything with type `forall a. a`, including `undefined`, `error "any string"`, `throw` and so on, has denotation ⊥.

66.4 Lazy pattern matching

You might have seen pattern matches like the following in Haskell sources.

```
-- From Control.Arrow
(***) f g ~(x, y) = (f x, g y)
```

The question is: what does the tilde sign (~) mean in the above pattern match? ~ makes a *lazy pattern* or *irrefutable pattern*. Normally, if you pattern match using a constructor as part of the pattern, you have to evaluate any argument passed into that function to make sure it matches the pattern. For example, if you had a function like the above, the third argument would be evaluated when you call the function to make sure the value matches the pattern. (Note that the first and second arguments won't be evaluated, because the patterns `f` and `g` match anything. Also it's worth noting that the *components* of the tuple won't be evaluated: just the 'top level'. Try `let f (Just x) = 1 in f (Just undefined)` to see this.)

However, prepending a pattern with a tilde sign delays the evaluation of the value until the component parts are actually used. But you run the risk that the value might not match the pattern — you're telling the compiler 'Trust me, I know it'll work out'. (If it turns out it doesn't match the pattern, you get a runtime error.) To illustrate the difference:

```
Prelude> let f (x,y) = 1
Prelude> f undefined
*** Exception: Prelude.undefined

Prelude> let f ~(x,y) = 1
Prelude> f undefined
1
```

[2] Chapter 58 on page 405

In the first example, the value is evaluated because it has to match the tuple pattern. You evaluate undefined and get undefined, which stops the proceedings. In the latter example, you don't bother evaluating the parameter until it's needed, which turns out to be never, so it doesn't matter you passed it `undefined`. To bring the discussion around in a circle back to `(***)`:

```
Prelude> (const 1 *** const 2) undefined
(1,2)
```

If the pattern weren't irrefutable, the example would have failed.

66.4.1 When does it make sense to use lazy patterns?

Essentially, use lazy patterns when you only have the single constructor for the type, e.g. tuples. Multiple equations won't work nicely with irrefutable patterns. To see this, let's examine what would happen were we to make `head` have an irrefutable pattern:

```
head' :: [a] -> a
head' ~[]     = undefined
head' ~(x:xs) = x
```

We're using one of these patterns for sure, and we need not evaluate even the top level of the argument until absolutely necessary, so we don't know whether it's an empty list or a cons cell. As we're using an *irrefutable* pattern for the first equation, this will match, and the function will always return undefined.

Exercises:

- Why won't changing the order of the equations to `head'` help here?
- If the first equation is changed to use an ordinary refutable pattern, will the behavior of `head'` still be different from that of `head`? If so, how?

66.5 Benefits of nonstrict semantics

Why make Haskell a nonstrict language in the first place? What advantages are there?

66.5.1 Separation of concerns without time penalty

Lazy evaluation encourages a kind of "what you see is what you get" mentality when it comes to coding. For example, let's say you wanted to find the lowest three numbers in a long list. In Haskell this is achieved extremely naturally: `take 3 (sort xs)`. However a naïve translation of that code in a strict language would be a very bad idea! It would involve computing the entire sorted list, then chopping off all but the first three elements. However, with lazy evaluation we stop once we've sorted the list up to the third element, so this natural definition turns out to be efficient (depending on the implementation of sort).

To give a second example, the function `isInfixOf`[3] from Data.List allows you to see if one string is a substring of another string. It's easily definable as:

```
isInfixOf :: Eq a => [a] -> [a] -> Bool
isInfixOf x y = any (isPrefixOf x) (tails y)
```

Again, this would be suicide in a strict language as computing all the tails of a list would be very time-consuming. However, here we only evaluate enough elements of each tail to determine whether it has prefix `x` or not, and stop and return `True` once we found one with prefix `x`.

There are many more examples along these lines.[4] So we can write code that looks and feels natural and doesn't incur any time penalty.

However, as always in Computer Science (and in life), a tradeoff exists (in particular, a space-time tradeoff). Using a thunk instead of a plain `Int` for a trivial calculation (like `2+2`) can be a waste. For more examples, see the page on ../Strictness[5].

66.5.2 Improved code reuse

Often code reuse is desireable.

For example, we'll take again (but in more detail) `isInfixOf`[6] from Data.List. Let's look at the full definition:

```
-- From the Prelude
or = foldr (||) False
any p = or . map p

-- From Data.List
isPrefixOf []     _      = True
isPrefixOf _      []     = False
isPrefixOf (x:xs) (y:ys) = x == y && isPrefixOf xs ys

tails []         = [[]]
tails xss@(_:xs) = xss : tails xs

-- Our function
isInfixOf :: Eq a => [a] -> [a] -> Bool
isInfixOf x y = any (isPrefixOf x) (tails y)
```

Where `any`, `isPrefixOf` and `tails` are the functions taken from the `Data.List` library. This function determines if its first parameter, `x` occurs as a subsequence of its second, `y`; when applied on `String`'s (i.e. `[Char]`), it checks if `x` is a substring of `y`. Read in a strict way, it forms the list of all the tails of `y`, then checks them all to see if any of them have `x` as a prefix. In a strict language, writing this function this way (relying on the already-written programs `any`, `isPrefixOf`, and `tails`) would be silly, because it would be far slower than it needed to be. You'd end up doing direct recursion again, or in an imperative language, a couple of nested loops. You might be able to get some use out of `isPrefixOf`, but you

[3] http://haskell.org/ghc/docs/latest/html/libraries/base/Data-List.html#v%3AisInfixOf

[4] In general, expressions like `prune . generate`, where `generate` produces a list of items and `prune` cuts them down, will be much more efficient in a nonstrict language.

[5] Chapter 67 on page 487

[6] http://haskell.org/ghc/docs/latest/html/libraries/base/Data-List.html#v%3AisInfixOf

certainly wouldn't use `tails`. You might be able to write a usable shortcutting `any`, but it would be more work, since you wouldn't want to use `foldr` to do it.

Now, in a lazy language, all the shortcutting is done for you. You don't end up rewriting `foldr` to shortcut when you find a solution or rewriting the recursion done in tails so that it will stop early again. You can reuse standard library code better. Laziness isn't just a constant-factor speed thing, it makes a qualitative impact on the code which is reasonable to write. In fact, it's commonplace to define infinite structures, and then only use as much as is needed, rather than having to mix up the logic of constructing the data structure with code that determines whether any part is needed. Code modularity is increased, as laziness gives you more ways to chop up your code into small pieces, each of which does a simple task of generating, filtering, or otherwise manipulating data.

Why Functional Programming Matters[7] largely focuses on examples where laziness is crucial and provides a strong argument for lazy evaluation being the default.

66.5.3 Infinite data structures

Examples:

```
fibs = 1:1:zipWith (+) fibs (tail fibs)
"rock-scissors-paper" example from Bird&Wadler
prune . generate
```

Infinite data structures usually tie a knot, too, but the Sci-Fi-Explanation of that is better left to the next section. One could move the next section before this one but I think that infinite data structures are simpler than tying general knots

66.6 Common nonstrict idioms

66.6.1 Tying the knot

More practical examples?

```
repMin
```

Sci-Fi-Explanation: "You can borrow things from the future as long as you don't try to change them". Advanced: the "Blueprint"-technique. Examples: the one from the haskellwiki, the one from the mailing list.

At first a pure functional language seems to have a problem with circular data structures. Suppose I have a data type like this:

```
data Foo a = Foo {value :: a, next :: Foo a}
```

If I want to create two objects "x" and "y" where "x" contains a reference to "y" and "y" contains a reference to "x" then in a conventional language this is straightforward: create

[7] http://www.cse.chalmers.se/~rjmh/Papers/whyfp.html

the objects and then set the relevant fields to point to each other:

```
-- Not Haskell code
x := new Foo;
y := new Foo;
x.value := 1;
x.next := y;
y.value := 2
y.next := x;
```

In Haskell this kind of modification is not allowed. So instead we depend on lazy evaluation:

```
circularFoo :: Foo Int
circularFoo = x
   where
      x = Foo 1 y
      y = Foo 2 x
```

This depends on the fact that the ”Foo” constructor is a function, and like most functions it gets evaluated lazily. Only when one of the fields is required does it get evaluated.

It may help to understand what happens behind the scenes here. When a lazy value is created (for example, by a call to ”Foo”), the compiler generates an internal data structure called a ”thunk” containing the function call and arguments. When the value of the function is demanded, the function is called (as you would expect). But then the thunk data structure is replaced with the return value. Thus, anything else that refers to that value gets it straight away without the need to call the function.

(Note that the Haskell language standard makes no mention of thunks: they are an implementation mechanism. From the mathematical point of view this is a straightforward example of mutual recursion.)

So, when we call ”circularFoo” the result ”x” is actually a thunk. One of the arguments is a reference to a second thunk representing ”y”. This in turn has a reference back to the thunk representing ”x”. If we then use the value ”next x” this forces the ”x” thunk to be evaluated and returns a reference to the ”y” thunk. If I use the value ”next $ next x” then I force the evaluation of both thunks. So now both thunks have been replaced with the actual ”Foo” structures, referring to each other. Which is what we wanted.

This is most often applied with constructor functions, but it isn't limited just to constructors. You can just as readily write:

```
x = f y
y = g x
```

The same logic applies.

66.6.2 Memoization, Sharing and Dynamic Programming

Dynamic programming with immutable arrays. DP with other finite maps, Hinze's paper ”Trouble shared is Trouble halved”. Let-floating \x-> let z = foo x in \y -> ....

66.7 Conclusions about laziness

<--! Move conclusions to the introduction? -->

Laziness:

- Can make qualitative improvements to performance!
- Can hurt performance in some other cases.
- Makes code simpler.
- Makes hard problems conceivable.
- Allows for separation of concerns with regard to generating and processing data.

66.8 References

- Laziness on the Haskell wiki[8]
- Lazy evaluation tutorial on the Haskell wiki[9]

[8] http://www.haskell.org/haskellwiki/Performance/Laziness

[9] http://www.haskell.org/haskellwiki/Haskell/Lazy_Evaluation

67 Strictness

67.1 Difference between strict and lazy evaluation

Strict evaluation, or eager evaluation, is an evaluation strategy where expressions are evaluated as soon as they are bound to a variable. For example, with strict evaluation, when `x = 3 * 7` is read, 3 * 7 is immediately computed and 21 is bound to x. Conversely, with lazy evaluation[1] values are only computed when they are needed. In the example `x = 3 * 7`, 3 * 7 isn't evaluated until it's needed, like if you needed to output the value of x.

67.2 Why laziness can be problematic

Lazy evaluation often involves objects called thunks. A thunk is a placeholder object, specifying not the data itself, but rather how to compute that data. An entity can be replaced with a thunk to compute that entity. When an entity is copied, whether or not it is a thunk doesn't matter - it's copied as is (on most implementations, a pointer to the data is created). When an entity is evaluated, it is first checked if it is thunk; if it's a thunk, then it is executed, otherwise the actual data is returned. It is by the magic of thunks that laziness can be implemented.

Generally, in the implementation the thunk is really just a pointer to a piece of (usually static) code, plus another pointer to the data the code should work on. If the entity computed by the thunk is larger than the pointer to the code and the associated data, then a thunk wins out in memory usage. But if the entity computed by the thunk is smaller, the thunk ends up using more memory.

As an example, consider an infinite length list generated using the expression `iterate (+ 1) 0`. The size of the list is infinite, but the code is just an add instruction, and the two pieces of data, 1 and 0, are just two Integers. In this case, the thunk representing that list takes much less memory than the actual list, which would take infinite memory.

However, as another example consider the number generated using the expression `4 * 13 + 2`. The value of that number is 54, but in thunk form it is a multiply, an add, and three numbers. In such a case, the thunk loses in terms of memory.

Often, the second case above will consume so much memory that it will consume the entire heap and force the garbage collector. This can slow down the execution of the program significantly. And that, in fact, is the main reason why laziness can be problematic.

[1] Chapter 66 on page 475

Additionally, if the resulting value *is* used, no computation is saved; instead, a slight overhead (of a constant factor) for building the thunk is paid. However, this overhead is not something the programmer should deal with most of the times; more important factors must be considered and may give a much bigger improvements; additionally, optimizing Haskell compilers like GHC can perform 'strictness analysis' and remove that slight overhead.

67.3 Strictness annotations

67.4 seq

67.4.1 DeepSeq

67.5 References

- Strictness on the Haskell wiki[2]

[2] `http://www.haskell.org/haskellwiki/Performance/Strictness`

68 Algorithm complexity

Complexity Theory is the study of how long a program will take to run, depending on the size of its input. There are many good introductory books to complexity theory and the basics are explained in any good algorithms book. I'll keep the discussion here to a minimum.

The idea is to say how well a program scales with more data. If you have a program that runs quickly on very small amounts of data but chokes on huge amounts of data, it's not very useful (unless you know you'll only be working with small amounts of data, of course). Consider the following Haskell function to return the sum of the elements in a list:

```
sum [] = 0
sum (x:xs) = x + sum xs
```

How long does it take this function to complete? That's a very difficult question; it would depend on all sorts of things: your processor speed, your amount of memory, the exact way in which the addition is carried out, the length of the list, how many other programs are running on your computer, and so on. This is far too much to deal with, so we need to invent a simpler model. The model we use is sort of an arbitrary ”machine step.” So the question is ”how many machine steps will it take for this program to complete?” In this case, it only depends on the length of the input list.

If the input list is of length 0, the function will take either 0 or 1 or 2 or some very small number of machine steps, depending exactly on how you count them (perhaps 1 step to do the pattern matching and 1 more to return the value 0). What if the list is of length 1? Well, it would take however much time the list of length 0 would take, plus a few more steps for doing the first (and only element).

If the input list is of length n, it will take however many steps an empty list would take (call this value y) and then, for each element it would take a certain number of steps to do the addition and the recursive call (call this number x). Then, the total time this function will take is $nx+y$ since it needs to do those additions n many times. These x and y values are called *constant*values*, since they are independent of* n*, and actually dependent*only on exactly how we define a machine step, so we really don't want to consider them all that important. Therefore, we say that the complexity of this `sum` function is $\mathcal{O}(n)$ (read ”order n”). Basically saying something is $\mathcal{O}(n)$ means that for some constant factors x and y, the function takes $nx+y$ machine steps to complete.

Consider the following sorting algorithm for lists (commonly called ”insertion sort”):

```
sort []  = []
sort [x] = [x]
sort (x:xs) = insert (sort xs)
    where insert [] = [x]
```

```
insert (y:ys) | x <= y    = x : y : ys
              | otherwise = y : insert ys
```

The way this algorithm works is as follow: if we want to sort an empty list or a list of just one element, we return them as they are, as they are already sorted. Otherwise, we have a list of the form `x:xs`. In this case, we sort `xs` and then want to insert `x` in the appropriate location. That's what the `insert`function does. It traverses the now-sorted tail and inserts `x`wherever it naturally fits.

Let's analyze how long this function takes to complete. Suppose it takes $f(n)$ stepts to sort a list of length n. Then, in order to sort a list of n-many elements, we first have to sort the tail of the list first, which takes $f(n-1)$ time. Then, we have to insert `x` into this new list. If `x` has to go at the end, this will take $\mathcal{O}(n-1) = \mathcal{O}(n)$ steps. Putting all of this together, we see that we have to do $\mathcal{O}(n)$ amount of work $\mathcal{O}(n)$ many times, which means that the entire complexity of this sorting algorithm is $\mathcal{O}(n^2)$. Here, the squared is not a constant value, so we cannot throw it out.

What does this mean? Simply that for really long lists, the `sum`function won't take very long, but that the `sort` function will take quite some time. Of course there are algorithms that run much more slowly than simply $\mathcal{O}(n^2)$ and there are ones that run more quickly than $\mathcal{O}(n)$. (Also note that a $\mathcal{O}(n^2)$ algorithm may actually be much faster than a $\mathcal{O}(n)$ algorithm in practice, if it takes much less time to perform a single step of the $\mathcal{O}(n^2)$ algorithm.)

Consider the random access functions for lists and arrays. In the worst case, accessing an arbitrary element in a list of length n will take $\mathcal{O}(n)$ time (think about accessing the last element). However with arrays, you can access any element immediately, which is said to be in *constant* time, or $\mathcal{O}(1)$, which is basically as fast an any algorithm can go.

There's much more in complexity theory than this, but this should be enough to allow you to understand all the discussions in this tutorial. Just keep in mind that $\mathcal{O}(1)$ is faster than $\mathcal{O}(n)$ is faster than $\mathcal{O}(n^2)$, etc.

68.1 Optimising

68.1.1 Profiling

69 Libraries Reference

70 The Hierarchical Libraries

1. REDIRECT Haskell/Libraries[1]

[1] https://en.wikibooks.org/wiki/Haskell%2FLibraries

71 Lists

The **List** datatype (see Data.List[1]) is the fundamental data structure in Haskell — this is the basic building-block of data storage and manipulation. In computer science terms it is a singly-linked list. In the hierarchical library system the List module is stored in `Data.List`; but this module only contains utility functions. The definition of list itself is integral to the Haskell language.

71.1 Theory

A singly-linked list is a set of values in a defined order. The list can only be traversed in one direction (i.e., you cannot move back and forth through the list like tape in a cassette machine).

The list of the first 5 positive integers is written as

```
[ 1, 2, 3, 4, 5 ]
```

We can move through this list, examining and changing values, from left to right, but not in the other direction. This means that the list

```
[ 5, 4, 3, 2, 1 ]
```

is not just a trivial change in perspective from the previous list, but the result of significant computation (*O(n)* in the length of the list).

71.2 Definition

The polymorphic list datatype can be defined with the following recursive definition:

```
data [a] = []
         | a : [a]
```

The ”base case” for this definition is `[]`, the empty list. In order to put something into this list, we use the `(:)` constructor

[1] `http://www.haskell.org/ghc/docs/latest/html/libraries/base/Data-List.html`

```
emptyList = []
oneElem = 1:[]
```

The (:) (pronounced *cons*) is right-associative, so that creating multi-element lists can be done like

```
manyElems = 1:2:3:4:5:[]
```

or even just

```
manyElems' = [1,2,3,4,5]
```

71.3 Basic list usage

71.3.1 Prepending

It's easy to hard-code lists without cons, but run-time list creation will use cons. For example, to push an argument onto a simulated stack, we would use:

```
push :: Arg -> [Arg] -> [Arg]
push arg stack = arg:stack
```

71.3.2 Pattern-matching

If we want to examine the top of the stack, we would typically use a peek function. We can try pattern-matching for this.

```
peek :: [Arg] -> Maybe Arg
peek [] = Nothing
peek (a:as) = Just a
```

The a before the *cons* in the pattern matches the head of the list. The as matches the tail of the list. Since we don't actually want the tail (and it's not referenced anywhere else in the code), we can tell the compiler this explicitly, by using a wild-card match, in the form of an underscore:

```
peek (a:_) = Just a
```

71.4 List utilities

FIXME: is this not covered in the chapter on list manipulation[2]*?*

71.4.1 Maps

71.4.2 Folds, unfolds and scans

71.4.3 Length, head, tail etc.

Category:Haskell/Not in book[3]

[2] Chapter 14 on page 85

[3] `https://en.wikibooks.org/wiki/Category%3AHaskell%2FNot%20in%20book`

72 Arrays

1. REDIRECT Haskell/Libraries/Arrays[1]

[1] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FArrays

73 Maybe

1. REDIRECT Haskell/Libraries/Maybe[1]

[1] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FMaybe

74 Maps

1. REDIRECT Haskell/Libraries/Maps[1]

[1] `https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FMaps`

75 IO

1. REDIRECT Haskell/Libraries/IO[1]

[1] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FIO

76 Random Numbers

1. REDIRECT Haskell/Libraries/Random[1]

[1] https://en.wikibooks.org/wiki/Haskell%2FLibraries%2FRandom

77 General Practices

78 Building a standalone application

1. REDIRECT Haskell/Standalone programs[1]

[1] https://en.wikibooks.org/wiki/Haskell%2FStandalone%20programs

79 Debugging

79.1 Debug prints with `Debug.Trace`

Debug prints are a common way to debug programs. In imperative languages, we can just sprinkle the code with print statements to standard output or to some as log file in order to track debug information (e.g. value of a particular variable, or some human-readable message). In Haskell, however, we cannot output any information other than through the IO monad; and we don't want to introduce that just for debugging.

To deal with this problem, the standard library provides the Debug.Trace[1]. That module exports a function called `trace` which provides a convenient way to attach debug print statements anywhere in a program. For instance, this program prints every argument passed to `fib` when not equal to 0 or 1:

```
module Main where
import Debug.Trace

fib :: Int -> Int
fib 0 = 0
fib 1 = 1
fib n = trace ("n: " ++ show n) $ fib (n - 1) + fib (n - 2)

main = putStrLn $ "fib 4: " ++ show (fib 4)
```

Below is the resulting output:

```
n: 4
n: 3
n: 2
n: 2
fib 4: 3
```

Also, `trace` makes it possible to trace execution steps of program; that is, which function is called first, second, etc. To do so, we annotate parts of functions we are interested in, like this:

```
module Main where
import Debug.Trace

factorial :: Int -> Int
factorial n | n == 0    = trace ("branch 1") 1
            | otherwise = trace ("branch 2") $ n * (factorial $ n - 1)

main = do
    putStrLn $ "factorial 6: " ++ show (factorial 6)
```

[1] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Debug-Trace.html

When a program annotated in such way is run, it will print the debug strings in the same order the annotated statements were executed. That output might help to locate errors in case of missing statements or similar things.

79.1.1 Some extra advice

As demonstrated above, `trace` can be used outside of the IO monad; and indeed its type signature...

```
trace :: String -> a -> a
```

...indicates that it is a pure function. Yet surely `trace` *is* doing IO while printing useful messages. What's going on? In fact, `trace` uses a dirty trick of sorts to circumvent the separation between IO and pure Haskell. That is reflected in the following disclaimer, found in the documentation for `trace`[2]:

> The trace function should *only* be used for debugging, or for monitoring execution. The function is not referentially transparent: its type indicates that it is a pure function but it has the side effect of outputting the trace message.

A common mistake in using `trace`: while trying to fit the debug traces into an existing function, one accidentally includes the value being evaluated in the message to be printed by `trace`; e.g. don't do anything like this:

```
let foo = trace ("foo = " ++ show foo) $ bar
in  baz
```

That leads to infinite recursion because trace message will be evaluated before bar expression which will lead to evaluation of foo in terms of trace message and bar again and trace message will be evaluated before bar and so forth to infinity. Instead of `show foo`, the correct trace message should have `show bar`:

```
let foo = trace ("foo = " ++ show bar) $ bar
in  baz
```

79.1.2 Useful idioms

A helper function that incorporates `show` can be convenient:

```
traceThis :: (Show a) => a -> a
traceThis x = trace (show x) x
```

In a similar vein, `Debug.Trace` defines a `traceShow` function, that ”prints” its first argument and evaluates to the second one:

```
traceShow :: (Show a) => a -> b -> b
traceShow = trace . show
```

Finally, a function `debug` like this one may prove handy as well:

[2] `http://hackage.haskell.org/packages/archive/base/latest/doc/html/Debug-Trace#v:trace.html`

```
debug = flip trace
```

This will allow you to write code like...

```
main = (1 + 2) `debug` "adding"
```

... making it easier to comment/uncomment debugging statements.

79.2 Incremental development with GHCi

79.3 Debugging with Hat

79.4 General tips

80 Testing

80.1 Quickcheck

Consider the following function:

```
getList = find 5 where
     find 0 = return []
     find n = do
       ch <- getChar
       if ch `elem` ['a'..'e'] then do
             tl <- find (n-1)
             return (ch : tl) else
           find n
```

How would we effectively test this function in Haskell? We'll use refactoring and QuickCheck.

80.1.1 Keeping things pure

The getList function is hard to test because getChar does IO out in the world, so there's no internal way to verify things. The other statements in our do block are all wrapped up with the IO.

Let's untangle our function so we can at least test the referentially transparent parts with QuickCheck. We can take advantage of lazy IO firstly, to avoid all the unpleasant low-level IO handling.

So the first step is to factor out the IO part of the function into a thin "skin" layer:

```
-- A thin monadic skin layer
getList :: IO [Char]
getList = fmap take5 getContents

-- The actual worker
take5 :: [Char] -> [Char]
take5 = take 5 . filter (`elem` ['a'..'e'])
```

80.1.2 Testing with QuickCheck

Now, we can test the 'guts' of the algorithm, the take5 function, in isolation. Let's use QuickCheck. First we need an Arbitrary instance for the Char type — that takes care of generating random Chars for us to test with. Restrict it to a range of nice chars just for simplicity:

```
import Data.Char
import Test.QuickCheck

instance Arbitrary Char where
    arbitrary     = choose ('\32', '\128')
    coarbitrary c = variant (ord c `rem` 4)
```

Let's fire up GHCi and try some generic properties (it's nice that we can use the QuickCheck testing framework directly from the Haskell REPL). An easy one first, a [Char] is equal to itself:

```
*A> quickCheck ((\s -> s == s) :: [Char] -> Bool)
OK, passed 100 tests.
```

What just happened? QuickCheck generated 100 random [Char] values, and applied our property, checking the result was True for all cases. QuickCheck *generated the test sets for us*!

A more interesting property now: reversing twice returns the identity:

```
*A> quickCheck ((\s -> (reverse.reverse) s == s) :: [Char] -> Bool)
OK, passed 100 tests.
```

Great!

80.1.3 Testing take5

The first step to testing with QuickCheck is to work out some properties that are true of the function, for all inputs. That is, we need to find *invariants*.

A simple invariant might be: $\forall s. length(take5 s) = 5$

So let's write that as a QuickCheck property:

```
\s -> length (take5 s) == 5
```

Which we can then run in QuickCheck as:

```
*A> quickCheck (\s -> length (take5 s) == 5)
Falsifiable, after 0 tests:
""
```

Ah! QuickCheck caught us out. If the input string contains less than 5 filterable characters, the resulting string will be no more than 5 characters long. So let's weaken the property a bit: $\forall s. length(take5 s) \leq 5$

That is, take5 returns a string of at most 5 characters long. Let's test this:

```
*A> quickCheck (\s -> length (take5 s) <= 5)
OK, passed 100 tests.
```

Good!

80.1.4 Another property

Another thing to check would be that the correct characters are returned. That is, for all returned characters, those characters are members of the set ['a','b','c','d','e'].

We can specify that as: $\forall s.\forall e.(e \in take5 s) \Rightarrow (e \in \{a,b,c,d,e\})$

And in QuickCheck:

```
*A> quickCheck (\s -> all (`elem` ['a'..'e']) (take5 s))
OK, passed 100 tests.
```

Excellent. So we can have some confidence that the function neither returns strings that are too long nor includes invalid characters.

80.1.5 Coverage

One issue with the default QuickCheck configuration, when testing [Char]: the standard 100 tests isn't enough for our situation. In fact, QuickCheck never generates a String greater than 5 characters long when using the supplied Arbitrary instance for Char! We can confirm this:

```
*A> quickCheck (\s -> length (take5 s) < 5)
OK, passed 100 tests.
```

QuickCheck wastes its time generating different Chars, when what we really need is longer strings. One solution to this is to modify QuickCheck's default configuration to test deeper:

```
deepCheck p = check (defaultConfig { configMaxTest = 10000}) p
```

This instructs the system to find at least 10000 test cases before concluding that all is well. Let's check that it is generating longer strings:

```
*A> deepCheck (\s -> length (take5 s) < 5)
Falsifiable, after 125 tests:
";:iD^*NNi~Y\\RegMob\DEL@krsx/=dcf7kub|EQi\DELD*"
```

We can check the test data QuickCheck is generating using the 'verboseCheck' hook. Here, testing on integers lists:

```
*A> verboseCheck (\s -> length s < 5)
0: []
1: [0]
2: []
3: []
4: []
5: [1,2,1,1]
6: [2]
7: [-2,4,-4,0,0]
Falsifiable, after 7 tests:
[-2,4,-4,0,0]
```

80.1.6 More information on QuickCheck

- http://haskell.org/haskellwiki/Introduction_to_QuickCheck
- http://haskell.org/haskellwiki/QuickCheck_as_a_test_set_generator

80.2 HUnit

Sometimes it is easier to give an example for a test than to define one from a general rule. HUnit provides a unit testing framework which helps you to do just this. You could also abuse QuickCheck by providing a general rule which just so happens to fit your example; but it's probably less work in that case to just use HUnit.

TODO: give an example of HUnit test, and a small tour of it

More details for working with HUnit can be found in its user's guide[1].

[1] http://hunit.sourceforge.net/HUnit-1.0/Guide.html

81 Packaging your software (Cabal)

A guide to the best practice for creating a new Haskell project or program.

81.1 Recommended tools

Almost all new Haskell projects use the following tools. Each is intrinsically useful, but using a set of common tools also benefits everyone by increasing productivity, and you're more likely to get patches.

81.1.1 Revision control

Use darcs[1], unless you have a specific reason not to, in which case use git[2]. If you don't like git, go back and look at darcs. It's written in Haskell, and it's used by many Haskell developers. See the wikibook Understanding darcs[3] to get started.

81.1.2 Build system

Use Cabal[4]. You should read at least the start of section 2 of the Cabal User's Guide[5].

81.1.3 Documentation

For libraries, use Haddock[6]. We recommend using recent versions of haddock (2.8 or above, as of December 2010).

81.1.4 Testing

Pure code can be tested using QuickCheck[7] or SmallCheck[8], impure code with HUnit[9].

[1] http://darcs.net

[2] http://git-scm.com

[3] https://en.wikibooks.org/wiki/Understanding%20darcs

[4] http://haskell.org/cabal

[5] http://haskell.org/cabal/users-guide/index.html

[6] http://haskell.org/haddock

[7] http://www.md.chalmers.se/~rjmh/QuickCheck/

[8] http://www.mail-archive.com/haskell@haskell.org/msg19215.html

[9] http://hunit.sourceforge.net/

To get started, try Haskell/Testing[10]. For a slightly more advanced introduction, Simple Unit Testing in Haskell[11] is a blog article about creating a testing framework for QuickCheck using some Template Haskell.

81.2 Structure of a simple project

The basic structure of a new Haskell project can be adopted from HNop[12], the minimal Haskell project. It consists of the following files, for the mythical project ”haq”.

- Haq.hs -- the main haskell source file
- haq.cabal -- the cabal build description
- Setup.hs -- build script itself
- _darcs or .git -- revision control
- README -- info
- LICENSE -- license

You can of course elaborate on this, with subdirectories and multiple modules.

Here is a transcript on how you’d create a minimal darcs-using and cabalised Haskell project, for the cool new Haskell program ”haq”, build it, install it and release.

The command tool ’cabal init’ automates all this for you, but it’s important that you understand all the parts first.

We will now walk through the creation of the infrastructure for a simple Haskell executable. Advice for libraries follows after.

81.2.1 Create a directory

Create somewhere for the source:

```
$ mkdir haq
$ cd haq
```

81.2.2 Write some Haskell source

Write your program:

```
$ cat > Haq.hs
--
-- Copyright (c) 2006 Don Stewart - http://www.cse.unsw.edu.au/~dons
-- GPL version 2 or later (see http://www.gnu.org/copyleft/gpl.html)
--
import System.Environment
```

[10] Chapter 80 on page 517

[11] http://blog.codersbase.com/2006/09/simple-unit-testing-in-haskell.html

[12] http://semantic.org/hnop/

```
-- 'main' runs the main program
main :: IO ()
main = getArgs >>= print . haqify . head

haqify s = "Haq! " ++ s
```

81.2.3 Stick it in darcs

Place the source under revision control:

```
$ darcs init
$ darcs add Haq.hs
$ darcs record
addfile ./Haq.hs
Shall I record this change? (1/?)  [ynWsfqadjkc], or ? for help: y
hunk ./Haq.hs 1
+--
+-- Copyright (c) 2006 Don Stewart - http://www.cse.unsw.edu.au/~dons
+-- GPL version 2 or later (see http://www.gnu.org/copyleft/gpl.html)
+--
+import System.Environment
+
+-- | 'main' runs the main program
+main :: IO ()
+main = getArgs >>= print . haqify . head
+
+haqify s = "Haq! " ++ s
Shall I record this change? (2/?)  [ynWsfqadjkc], or ? for help: y
What is the patch name? Import haq source
Do you want to add a long comment? [yn]n
Finished recording patch 'Import haq source'
```

And we can see that darcs is now running the show:

```
$ ls
Haq.hs _darcs
```

For git:

```
$ git config --global user.name "John Doe"
$ git config --global user.email johndoe@example.com
$ git init
$ git add *
$ git commit -m 'Import haq source'
$ ls -A
.git Haq.hs
```

81.2.4 Add a build system

Create a .cabal file describing how to build your project:

```
$ cat > haq.cabal
Name:                haq
Version:             0.0
Synopsis:            Super cool mega lambdas
```

```
Description:        My super cool, indeed, even mega lambdas
                    will demonstrate a basic project. You will marvel.
License:            GPL
License-file:       LICENSE
Author:             Don Stewart
Maintainer:         Don Stewart <dons@cse.unsw.edu.au>
Build-Depends:      base

Executable:         haq
Main-is:            Haq.hs
```

(If your package uses other packages, e.g. `array`, you'll need to add them to the `Build-Depends:` field.) Add a `Setup.lhs` that will actually do the building:

```
$ cat > Setup.lhs
#! /usr/bin/env runhaskell

> import Distribution.Simple
> main = defaultMain
```

Cabal allows either `Setup.hs` or `Setup.lhs`; as long as the format is appropriate, it doesn't matter which one you choose. But it's a good idea to always include the `#! /usr/bin/env runhaskell` line; because it follows the shebang[13] convention, you could execute the Setup.hs directly in a Unix shell instead of always manually calling `runhaskell` (assuming the Setup file is marked executable, of course).

Record your changes:

```
$ darcs add haq.cabal Setup.lhs
$ darcs record --all
What is the patch name? Add a build system
Do you want to add a long comment? [yn]n
Finished recording patch 'Add a build system'
```

Git:

```
$ git add haq.cabal Setup.lhs
$ git commit -m 'Add a build system'
```

81.2.5 Build your project

Now build it!

```
$ runhaskell Setup.lhs configure --prefix=$HOME --user
$ runhaskell Setup.lhs build
$ runhaskell Setup.lhs install
```

[13] https://en.wikipedia.org/wiki/Shebang%20%28Unix%29

81.2.6 Run it

And now you can run your cool project:

```
$ haq me
"Haq! me"
```

You can also run it in-place, avoiding the install phase:

```
$ dist/build/haq/haq you
"Haq! you"
```

81.2.7 Build some haddock documentation

Generate some API documentation into dist/doc/*

```
$ runhaskell Setup.lhs haddock
```

which generates files in dist/doc/ including:

```
$ w3m -dump dist/doc/html/haq/Main.html
 haq Contents Index
 Main

 Synopsis
 main :: IO ()

 Documentation

 main :: IO ()
 main runs the main program

 Produced by Haddock version 0.7
```

No output? Make sure you have actually installed haddock. It is a separate program, not something that comes with the Haskell compiler, like Cabal.

81.2.8 Add some automated testing: QuickCheck

We'll use QuickCheck to specify a simple property of our Haq.hs code. Create a tests module, Tests.hs, with some QuickCheck boilerplate:

```
$ cat > Tests.hs
import Char
import List
import Test.QuickCheck
import Text.Printf

main  = mapM_ (\(s,a) -> printf "%-25s: " s >> a) tests

instance Arbitrary Char where
    arbitrary     = choose ('\0', '\128')
    coarbitrary c = variant (ord c `rem` 4)
```

Now let's write a simple property:

```
$ cat >> Tests.hs
-- reversing twice a finite list, is the same as identity
prop_reversereverse s = (reverse . reverse) s == id s
    where _ = s :: [Int]

-- and add this to the tests list
tests  = [("reverse.reverse/id", test prop_reversereverse)]
```

We can now run this test, and have QuickCheck generate the test data:

```
$ runhaskell Tests.hs
reverse.reverse/id       : OK, passed 100 tests.
```

Let's add a test for the 'haqify' function:

```
-- Dropping the "Haq! " string is the same as identity
prop_haq s = drop (length "Haq! ") (haqify s) == id s
    where haqify s = "Haq! " ++ s

tests  = [("reverse.reverse/id", test prop_reversereverse)
        ,("drop.haq/id",        test prop_haq)]
```

and let's test that:

```
$ runhaskell Tests.hs
reverse.reverse/id       : OK, passed 100 tests.
drop.haq/id              : OK, passed 100 tests.
```

Great!

81.2.9 Running the test suite from darcs

We can arrange for darcs to run the test suite on every commit:

```
$ darcs setpref test "runhaskell Tests.hs"
Changing value of test from '' to 'runhaskell Tests.hs'
```

will run the full set of QuickChecks. (If your test requires it you may need to ensure other things are built too e.g.: `darcs setpref test "alex Tokens.x;happy Grammar.y;runhaskell Tests.hs"`).

Let's commit a new patch:

```
$ darcs add Tests.hs
$ darcs record --all
What is the patch name? Add testsuite
Do you want to add a long comment? [yn]n
Running test...
reverse.reverse/id       : OK, passed 100 tests.
```

```
drop.haq/id              : OK, passed 100 tests.
Test ran successfully.
Looks like a good patch.
Finished recording patch 'Add testsuite'
```

Excellent, now patches must pass the test suite before they can be committed.

81.2.10 Tag the stable version, create a tarball, and sell it!

Tag the stable version:

```
$ darcs tag
What is the version name? 0.0
Finished tagging patch 'TAG 0.0'
```

Advanced Darcs functionality: lazy get

As your repositories accumulate patches, new users can become annoyed at how long it takes to accomplish the initial `darcs get`. (Some projects, like yi[14] or GHC, can have thousands of patches.) Darcs is quick enough, but downloading thousands of individual patches can still take a while. Isn't there some way to make things more efficient?

Darcs provides the `--lazy` option to `darcs get`. This enables to download only the latest version of the repository. Patches are later downloaded on demand if needed.

Distribution

When distributing your Haskell program, you have roughly three options:

1. distributing via a Darcs repository
2. distributing a tarball
    a) a Darcs tarball
    b) a Cabal tarball

With a Darcs repository, if it is public, then you are done. However: perhaps you don't have a server with Darcs, or perhaps your computer isn't set up for people to `darcs pull` from it. In which case you'll need to distribute the source via tarball.

Tarballs via darcs

Darcs provides a command where it will make a compressed tarball, and it will place a copy of all the files it manages into it. (Note that nothing in _darcs will be included - it'll just be your source files, no revision history.)

```
$ darcs dist -d haq-0.0
```

[14] `https://en.wikipedia.org/wiki/Yi%20%28editor%29`

```
Created dist as haq-0.0.tar.gz
```

And you're all set up!

Tarballs via Cabal

Since our code is cabalised, we can create a tarball with Cabal directly:

```
$ runhaskell Setup.lhs sdist
Building source dist for haq-0.0...
Source tarball created: dist/haq-0.0.tar.gz
```

This has advantages and disadvantages compared to a Darcs-produced tarball. The primary *advantage* is that Cabal will do more checking of our repository, and more importantly, it'll ensure that the tarball has the structure needed by HackageDB and cabal-install.

However, it does have a disadvantage: it packages up only the files needed to build the project. It will deliberately fail to include other files in the repository, even if they turn out to be necessary at some point[15]. To include other files (such as `Test.hs` in the above example), we need to add lines to the cabal file like:

```
extra-source-files: Tests.hs
```

If we had them, we could make sure files like AUTHORS or the README get included as well:

```
data-files: AUTHORS, README
```

81.2.11 Summary

The following files were created:

```
    $ ls
    Haq.hs          Tests.hs        dist            haq.cabal
    Setup.lhs       _darcs          haq-0.0.tar.gz
```

81.3 Libraries

The process for creating a Haskell library is almost identical. The differences are as follows, for the hypothetical "ltree" library:

[15] This is actually a good thing, since it allows us to do things like create an elaborate test suite which doesn't get included in the tarball, so users aren't bothered by it. It also can reveal hidden assumptions and omissions in our code - perhaps your code was only building and running because of a file accidentally generated.

81.3.1 Hierarchical source

The source should live under a directory path that fits into the existing module layout guide[16]. So we would create the following directory structure, for the module Data.LTree:

```
   $ mkdir Data
   $ cat > Data/LTree.hs
   module Data.LTree where
```

So our Data.LTree module lives in Data/LTree.hs

81.3.2 The Cabal file

Cabal files for libraries list the publically visible modules, and have no executable section:

```
   $ cat ltree.cabal
   Name:                ltree
   Version:             0.1
   Description:         Lambda tree implementation
   License:             BSD3
   License-file:        LICENSE
   Author:              Don Stewart
   Maintainer:          dons@cse.unsw.edu.au
   Build-Depends:       base
   Exposed-modules:     Data.LTree
```

We can thus build our library:

```
   $ runhaskell Setup.lhs configure --prefix=$HOME --user
   $ runhaskell Setup.lhs build
   Preprocessing library ltree-0.1...
   Building ltree-0.1...
   [1 of 1] Compiling Data.LTree       ( Data/LTree.hs, dist/build/Data/LTree.o
)
   /usr/bin/ar: creating dist/build/libHSltree-0.1.a
```

and our library has been created as a object archive. On *nix systems, you should probably add the --user flag to the configure step (this means you want to update your local package database during installation). Now install it:

```
   $ runhaskell Setup.lhs install
   Installing: /home/dons/lib/ltree-0.1/ghc-6.6 & /home/dons/bin ltree-0.1...
   Registering ltree-0.1...
   Reading package info from ".installed-pkg-config" ... done.
   Saving old package config file... done.
   Writing new package config file... done.
```

And we're done! You can use your new library from, for example, ghci:

[16] http://community.haskell.org/~simonmar/lib-hierarchy.html

```
$ ghci -package ltree
Prelude> :m + Data.LTree
Prelude Data.LTree>
```

The new library is in scope, and ready to go.

81.3.3 More complex build systems

For larger projects it is useful to have source trees stored in subdirectories. This can be done simply by creating a directory, for example, ”src”, into which you will put your src tree.

To have Cabal find this code, you add the following line to your Cabal file:

```
hs-source-dirs: src
```

Cabal can set up to also run configure scripts, along with a range of other features. For more information consult the Cabal documentation[17].

Internal modules

If your library uses internal modules that are not exposed, do not forget to list them in the *other-modules* field:

```
other-modules: My.Own.Module
```

Failing to do so (as of GHC 6.8.3) may lead to your library deceptively building without errors but actually being unusable from applications, which would fail at build time with a linker error.

81.4 Automation

81.4.1 cabal init

A package management tool for Haskell called cabal-install provides a command line tool to help developers create a simple cabal project. Just run and answer all the questions. Default values are provided for each.

```
$ cabal init
Package name [default "test"]?
Package version [default "0.1"]?
Please choose a license:
...
```

[17] http://www.haskell.org/ghc/docs/latest/html/Cabal/index.html

81.4.2 mkcabal

mkcabal is a tool that existed before cabal init, which also automatically populates a new cabal project :

```
    darcs get http://code.haskell.org/~dons/code/mkcabal
```

N.B. This tool does not work in Windows. The Windows version of GHC does not include the readline package that this tool needs.

Usage is:

```
$ mkcabal
Project name: haq
What license ["GPL","LGPL","BSD3","BSD4","PublicDomain","AllRightsReserved"]
 ["BSD3"]:
What kind of project [Executable,Library] [Executable]:
Is this your name? - "Don Stewart " [Y/n]:
Is this your email address? - "<dons@cse.unsw.edu.au>" [Y/n]:
Created Setup.lhs and haq.cabal
$ ls
Haq.hs    LICENSE   Setup.lhs _darcs    dist      haq.cabal
```

which will fill out some stub Cabal files for the project 'haq'.

To create an entirely new project tree:

```
$ mkcabal --init-project
Project name: haq
What license ["GPL","LGPL","BSD3","BSD4","PublicDomain","AllRightsReserved"]
 ["BSD3"]:
What kind of project [Executable,Library] [Executable]:
Is this your name? - "Don Stewart " [Y/n]:
Is this your email address? - "<dons@cse.unsw.edu.au>" [Y/n]:
Created new project directory: haq
$ cd haq
$ ls
Haq.hs    LICENSE   README    Setup.lhs haq.cabal
```

81.5 Licenses

Code for the common base library package must be BSD licensed or something more Free/Open. Otherwise, it is entirely up to you as the author.

Choose a licence (inspired by this[18]). Check the licences of things you use, both other Haskell packages and C libraries, since these may impose conditions you must follow.

Use the same licence as related projects, where possible. The Haskell community is split into 2 camps, roughly, those who release everything under BSD or public domain, and the GPL/LGPLers (this split roughly mirrors the copyleft/noncopyleft divide in Free software

[18] http://www.dina.dk/~abraham/rants/license.html

communities). Some Haskellers recommend specifically avoiding the LGPL, due to cross module optimisation issues. Like many licensing questions, this advice is controversial. Several Haskell projects (wxHaskell, HaXml, etc.) use the LGPL with an extra permissive clause to avoid the cross-module optimisation problem.

81.6 Releases

It's important to release your code as stable, tagged tarballs. Don't just rely on darcs for distribution[19].

- **darcs dist** generates tarballs directly from a darcs repository

For example:

```
$ cd fps
$ ls
Data      LICENSE   README    Setup.hs  TODO      _darcs    cbits dist
fps.cabal tests
$ darcs dist -d fps-0.8
Created dist as fps-0.8.tar.gz
```

You can now just post your fps-0.8.tar.gz

You can also have darcs do the equivalent of 'daily snapshots' for you by using a post-hook.

put the following in _darcs/prefs/defaults:

```
 apply posthook darcs dist
 apply run-posthook
```

Advice:

- Tag each release using **darcs tag**. For example:

```
$ darcs tag 0.8
Finished tagging patch 'TAG 0.8'
```

Then people can `darcs get --lazy --tag 0.8`, to get just the tagged version (and not the entire history).

81.7 Hosting

You can host public and private Darcs repositories on `http://patch-tag.com/` for free. Otherwise, a Darcs repository can be published simply by making it available from a web page. Another option is to host on the Haskell Community Server at `http://code.`

[19] `http://web.archive.org/web/20070627103346/http://awayrepl.blogspot.com/2006/11/we-dont-do-releases.html`

haskell.org/. You can request an account via http://community.haskell.org/admin/. You can also use https://github.com/ for Git hosting.

81.8 Example

A complete example[20] of writing, packaging and releasing a new Haskell library under this process has been documented.

[20] http://www.cse.unsw.edu.au/~dons/blog/2006/12/11#release-a-library-today

82 Using the Foreign Function Interface (FFI)

Using Haskell is fine, but in the real world there are a large number of useful libraries in other languages, especially C. To use these libraries, and let C code use Haskell functions, there is the Foreign Function Interface (FFI).

82.1 Calling C from Haskell

82.1.1 Marshalling (Type Conversion)

When using C functions, it is necessary to convert Haskell types to the appropriate C types. These are available in the `Foreign.C.Types` module; some examples are given in the following table.

Haskell	Foreign.C.Types	C
Double	CDouble	double
Char	CUChar	unsigned char
Int	CLong	long int

The operation of converting Haskell types into C types is called **marshalling** (and the opposite, predictably, *unmarshalling*). For basic types this is quite straightforward: for floating-point one uses `realToFrac` (either way, as e.g. both `Double` and `CDouble` are instances of classes `Real` and `Fractional`), for integers `fromIntegral`, and so on.

⚠ **Warning**

If you are using GHC previous to 6.12.x, note that the `CLDouble` type does not really represent a `long double`, but is just a synonym for `CDouble`: *never use it*, since it will lead to silent type errors if the C compiler does not also consider `long double` a synonym for `double`. Since 6.12.x `CLDouble` has been removed[1], pending proper implementation[2].

82.1.2 Calling a pure C function

A pure function implemented in C does not present significant trouble in Haskell. The `sin` function of the C standard library is a fine example:

```
{-# LANGUAGE ForeignFunctionInterface #-}

import Foreign
import Foreign.C.Types

foreign import ccall unsafe "math.h sin"
     c_sin :: CDouble -> CDouble
```

First, we specify a GHC extension for the FFI in the first line. We then import the Foreign and Foreign.C.Types modules, the latter of which contains information about CDouble, the representation of double-precision floating-point numbers in C.

We then specify that we are *import*ing a *foreign* function, with a call to C. A "safety level" has to be specified with the keyword safe (the default) or unsafe. In general, unsafe is more efficient, and safe is required only for C code that could call back a Haskell function. Since that is a very particular case, it is actually quite safe to use the unsafe keyword in most cases. Finally, we need to specify header and function name, separated by a space.

The Haskell function name is then given, in our case we use a standard c_sin, but it could have been anything. Note that the function signature must be correct—GHC will not check the C header to confirm that the function actually takes a CDouble and returns another, and writing a wrong one could have unpredictable results.

It is then possible to generate a wrapper around the function using CDouble so that it looks exactly like any Haskell function.

```
haskellSin :: Double -> Double
haskellSin = realToFrac . c_sin . realToFrac
```

Importing C's sin is simple because it is a pure function that takes a plain double as input and returns another as output: things will complicate with impure functions and pointers, which are ubiquitous in more complicated C libraries.

82.1.3 Impure C Functions

A classic impure C function is rand, for the generation of pseudo-random numbers. Suppose you do not want to use Haskell's System.Random.randomIO, for example because you want to replicate exactly the series of pseudo-random numbers output by some C routine. Then, you could import it just like sin before:

```
{-# LANGUAGE ForeignFunctionInterface #-}

import Foreign
import Foreign.C.Types

foreign import ccall unsafe "stdlib.h rand"
     c_rand :: CUInt -- Oops!
```

If you try this naïve implementation in GHCI, you will notice that c_rand is returning always the same value:

```
> c_rand
1714636915
> c_rand
1714636915
```

indeed, we have told GHC that it is a pure function, and GHC sees no point in calculating twice the result of a pure function. Note that GHC did not give any error or warning message.

In order to make GHC understand this is no pure function, we have to use the IO monad[3]:

```
{-# LANGUAGE ForeignFunctionInterface #-}

import Foreign
import Foreign.C.Types

foreign import ccall unsafe "stdlib.h rand"
     c_rand :: IO CUInt

foreign import ccall "stdlib.h srand"
     c_srand :: CUInt -> IO ()
```

Here, we also imported the `srand` function, to be able to seed the C pseudo-random generator.

```
> c_rand
1957747793
> c_rand
424238335
> c_srand 0
> c_rand
1804289383
> c_srand 0
> c_rand
1804289383
```

82.1.4 Working with C Pointers

The most useful C functions are often those that do complicated calculations with several parameters, and with increasing complexity the need of returning control codes arises. This means that a typical paradigm of C libraries is to give pointers of allocated memory as ”targets” in which the results may be written, while the function itself returns an integer value (typically, if 0, computation was successful, otherwise there was a problem specified by the number). Another possibility is that the function will return a pointer to a structure (possibly defined in the implementation, and therefore unavailable to us).

As a pedagogical example, we consider the `gsl_frexp` function[4] of the GNU Scientific Library[5], a freely available library for scientific computation. It is a simple C function with prototype:

[3] Chapter 34 on page 199

[4] http://www.gnu.org/software/gsl/manual/html_node/Elementary-Functions.html

[5] http://en.wikipedia.org/wiki/GNU_Scientific_Library

```
double gsl_frexp (double x, int * e)
```

The function takes a `double` x, and it returns its normalised fraction f and integer exponent e so that:

$$x = f \times 2^e \qquad e \in \mathbb{Z}, \quad 0.5 \leq f < 1$$

We interface this C function into Haskell with the following code:

```
{-# LANGUAGE ForeignFunctionInterface #-}

import Foreign
import Foreign.Ptr
import Foreign.C.Types

foreign import ccall unsafe "gsl/gsl_math.h gsl_frexp"
     gsl_frexp :: CDouble -> Ptr CInt -> IO CDouble
```

The new part is `Ptr`, which can be used with any instance of the `Storable` class, among which all C types, but also several Haskell types.

Notice how the result of the `gsl_frexp` function is in the `IO` monad. This is typical when working with pointers, be they used for input or output (as in this case); we will see shortly what would happen had we used a simple `CDouble` for the function.

The `frexp` function is implemented in pure Haskell code as follows:

```
frexp :: Double -> (Double, Int)
frexp x = unsafePerformIO $
    alloca $ \expptr -> do
        f <- gsl_frexp (realToFrac x) expptr
        e <- peek expptr
        return (realToFrac f, fromIntegral e)
```

We know that, memory management details aside, the function is pure: that's why the signature returns a tuple with f and e outside of the `IO` monad. Yet, f is provided *inside* of it: to extract it, we use the function *unsafePerformIO*, which extracts values from the `IO` monad: obviously, it is legitimate to use it only when we *know* the function is pure, and we can allow GHC to optimise accordingly.

To allocate pointers, we use the `alloca` function, which also takes responsibility for freeing memory. As an argument, `alloca` takes a function of type `Ptr a -> IO b`, and returns the `IO b`. In practice, this translates to the following usage pattern with λ functions:

```
... alloca $ \pointer -> do
        c_function argument pointer
        result <- peek pointer
        return result
```

The pattern can easily be nested if several pointers are required:

```
... alloca $ \firstPointer ->
        alloca $ \secondPointer -> do
            c_function argument firstPointer secondPointer
            first  <- peek firstPointer
            second <- peek secondPointer
            return (first, second)
```

Back to our `frexp` function: in the λ function that is the argument to `alloca`, the function is evaluated and the pointer is read immediately afterwards with `peek`. Here we can understand why we wanted the imported C function `gsl_frexp` to return a value in the `IO` monad: if GHC could decide when to calculate the quantity *f*, it would likely decide not to do it until it is necessary: that is at the last line when `return` uses it, and *after e* has been read from an allocated, but yet uninitialised memory address, which will contain random data. In short, we want `gsl_frexp` to return a monadic value because we want to determine the sequence of computations ourselves.

If some other function had required a pointer to *provide input* instead of storing output, one would have used the similar `poke` function to set the pointed value, obviously *before* evaluating the function:

```
... alloca $ \inputPointer ->
        alloca $ \outputPointer -> do
            poke inputPointer value
            c_function argument inputPointer outputPointer
            result <- peek outputPointer
            return result
```

In the final line, the results are arranged in a tuple and returned, after having been converted from C types.

To test the function, remember to link GHC to the GSL; in GHCI, do:

```
$ ghci frexp.hs -lgsl
```

(Note that most systems do not come with the GSL preinstalled, and you may have to download and install its development packages.)

82.1.5 Working with C Structures

Very often data are returned by C functions in form of `struct`s or pointers to these. In some rare cases, these structures are returned directly, but more often they are returned as pointers; the return value is most often an `int` that indicates the correctness of execution.

We will consider another GSL function, `gsl_sf_bessel_Jn_e`[6]. This function provides the regular cylindrical Bessel function for a given order *n*, and returns the result as a `gsl_sf_result` structure pointer. The structure contains two `double`s, one for the result and one for the error. The integer error code returned by the function can be transformed in a C string by the function `gsl_strerror`. The signature of the Haskell function we are looking for is therefore:

```
BesselJn :: Int -> Double -> Either String (Double, Double)
```

[6] http://www.gnu.org/software/gsl/manual/html_node/Regular-Cylindrical-Bessel-Functions.html

where the first argument is the order of the cylindrical Bessel function, the second is the function's argument, and the returned value is either an error message or a tuple with result and margin of error.

Making a New Instance of the `Storable` class

In order to allocate and read a pointer to a `gsl_sf_result` structure, it is necessary to make it an instance of the `Storable` class.

In order to do that, it is useful to use the `hsc2hs` program: we create first a `Bessel.hsc` file, with a mixed syntax of Haskell and C macros, which is later expanded into Haskell by the command:

```
$ hsc2hs Bessel.hsc
```

After that, we simply load the `Bessel.hs` file in GHC.

This is the first part of file `Bessel.hsc`:

```haskell
{-# LANGUAGE ForeignFunctionInterface #-}

module Bessel (besselJn) where

import Foreign
import Foreign.Ptr
import Foreign.C.String
import Foreign.C.Types

#include <gsl/gsl_sf_result.h>

data GslSfResult = GslSfResult { gsl_value :: CDouble, gsl_error :: CDouble }

instance Storable GslSfResult where
    sizeOf    _ = (#size gsl_sf_result)
    alignment _ = alignment (undefined :: CDouble)
    peek ptr = do
        value <- (#peek gsl_sf_result, val) ptr
        error <- (#peek gsl_sf_result, err) ptr
        return  GslSfResult { gsl_value = value, gsl_error = error }
    poke ptr (GslSfResult value error) = do
        (#poke gsl_sf_result, val) ptr value
        (#poke gsl_sf_result, err) ptr error
```

We use the `#include` directive to make sure `hsc2hs` knows where to find information about `gsl_sf_result`. We then define a Haskell data structure mirroring the GSL's, with two `CDouble`s: this is the class we make an instance of `Storable`. Strictly, we need only `sizeOf`, `alignment` and `peek` for this example; `poke` is added for completeness.

- `sizeOf` is obviously fundamental to the allocation process, and is calculated by `hsc2hs` with the `#size` macro.
- `alignment` is the size in bytes of the data structure alignment[7]. In general, it should be the largest `alignment` of the elements of the structure; in our case, since the two elements

[7] http://en.wikipedia.org/wiki/Data_structure_alignment

are the same, we simply use `CDouble`'s. The value of the argument to `alignment` is inconsequential, what is important is the type of the argument.

- `peek` is implemented using a `do`-block and the `#peek` macros, as shown. `val` and `err` are the names used for the structure fields in the GSL source code.
- Similarly, `poke` is implemented with the `#poke` macro.

Importing the C Functions

```
foreign import ccall unsafe "gsl/gsl_bessel.h gsl_sf_bessel_Jn_e"
     c_besselJn :: CInt -> CDouble -> Ptr GslSfResult -> IO CInt

foreign import ccall unsafe "gsl/gsl_errno.h gsl_set_error_handler_off"
     c_deactivate_gsl_error_handler :: IO ()

foreign import ccall unsafe "gsl/gsl_errno.h gsl_strerror"
     c_error_string :: CInt -> IO CString
```

We import several functions from the GSL libraries: first, the Bessel function itself, which will do the actual work. Then, we need a particular function, `gsl_set_error_handler_off`, because the default GSL error handler will simply crash the program, even if called by Haskell: we, instead, plan to deal with errors ourselves. The last function is the GSL-wide interpreter that translates error codes in human-readable C strings.

Implementing the Bessel Function

Finally, we can implement the Haskell version of the GSL cylindrical Bessel function of order *n*.

```
besselJn :: Int -> Double -> Either String (Double, Double)
besselJn n x = unsafePerformIO $
    alloca $ \gslSfPtr -> do
        c_deactivate_gsl_error_handler
        status <- c_besselJn (fromIntegral n) (realToFrac x) gslSfPtr
        if status == 0
            then do
                GslSfResult val err <- peek gslSfPtr
                return $ Right (realToFrac val, realToFrac err)
            else do
                error <- c_error_string status
                error_message <- peekCString error
                return $ Left ("GSL error: "++error_message)
```

Again, we use `unsafePerformIO` because the function is pure, even though its nuts-and-bolts implementation is not. After allocating a pointer to a GSL result structure, we deactivate the GSL error handler to avoid crashes in case something goes wrong, and finally we can call the GSL function. At this point, if the `status` returned by the function is 0, we unmarshal the result and return it as a tuple. Otherwise, we call the GSL error-string function, and pass the error as a `Left` result instead.

Examples

Once we are finished writing the `Bessel.hsc` function, we have to convert it to proper Haskell and load the produced file:

```
$ hsc2hs Bessel.hsc
$ ghci Bessel.hs -lgsl
```

We can then call the Bessel function with several values:

```
> besselJn 0 10
Right (-0.2459357644513483,1.8116861737200453e-16)
> besselJn 1 0
Right (0.0,0.0)
> besselJn 1000 2
Left "GSL error: underflow"
```

82.1.6 Advanced Topics

This section contains an advanced example with some more complex features of the FFI. We will import into Haskell one of the more complicated functions of the GSL, the one used to calculate the integral of a function between two given points with an adaptive Gauss-Kronrod algorithm[8]. The GSL function is `gsl_integration_qag`.

This example will illustrate function pointers, export of Haskell functions to C routines, enumerations, and handling pointers of unknown structures.

Available C Functions and Structures

The GSL has three functions which are necessary to integrate a given function with the considered method:

```
gsl_integration_workspace * gsl_integration_workspace_alloc (size_t n);
void gsl_integration_workspace_free (gsl_integration_workspace * w);
int gsl_integration_qag (const gsl_function * f, double a, double b,
                         double epsabs, double epsrel, size_t limit,
                         int key, gsl_integration_workspace * workspace,
                         double * result, double * abserr);
```

The first two deal with allocation and deallocation of a ”workspace” structure of which we know nothing (we just pass a pointer around). The actual work is done by the last function, which requires a pointer to a workspace.

To provide functions, the GSL specifies an appropriate structure for C:

```
struct gsl_function
{
  double (* function) (double x, void * params);
```

[8] http://www.gnu.org/software/gsl/manual/html_node/QAG-adaptive-integration.html

```
  void * params;
};
```

The reason for the `void` pointer is that it is not possible to define λ functions in C: parameters are therefore passed along with a parameter of unknown type. In Haskell, we do not need the `params` element, and will consistently ignore it.

Imports and Inclusions

We start our `qag.hsc` file with the following:

```
{-# LANGUAGE ForeignFunctionInterface, EmptyDataDecls #-}

module Qag ( qag,
             gauss15,
             gauss21,
             gauss31,
             gauss41,
             gauss51,
             gauss61 ) where

import Foreign
import Foreign.Ptr
import Foreign.C.Types
import Foreign.C.String

#include <gsl/gsl_math.h>
#include <gsl/gsl_integration.h>

foreign import ccall unsafe "gsl/gsl_errno.h gsl_strerror"
     c_error_string :: CInt -> IO CString

foreign import ccall unsafe "gsl/gsl_errno.h gsl_set_error_handler_off"
    c_deactivate_gsl_error_handler :: IO ()
```

We declare the `EmptyDataDecls` pragma, which we will use later for the `Workspace` data type. Since this file will have a good number of functions that should not be available to the outside world, we also declare it a module and export only the final function `qag` and the `gauss-` flags. We also include the relevant C headers of the GSL. The import of C functions for error messages and deactivation of the error handler was described before.

Enumerations

One of the arguments of `gsl_integration_qag` is `key`, an integer value that can have values from 1 to 6 and indicates the integration rule. GSL defines a macro for each value, but in Haskell it is more appropriate to define a type, which we call `IntegrationRule`. Also, to have its values automatically defined by `hsc2hs`, we can use the `enum` macro:

```
newtype IntegrationRule = IntegrationRule { rule :: CInt }
#{enum IntegrationRule, IntegrationRule,
    gauss15 = GSL_INTEG_GAUSS15,
    gauss21 = GSL_INTEG_GAUSS21,
    gauss31 = GSL_INTEG_GAUSS31,
    gauss41 = GSL_INTEG_GAUSS41,
    gauss51 = GSL_INTEG_GAUSS51,
    gauss61 = GSL_INTEG_GAUSS61
  }
```

hsc2hs will search the headers for the macros and give our variables the correct values. The enum directive will define a function with an appropriate type signature for each of the enum values. The above example will get translated to something like this (with the C macros appropriately replaced by their values):

```
newtype IntegrationRule = IntegrationRule { rule :: CInt }

gauss15  :: IntegrationRule
gauss15  = IntegrationRule GSL_INTEG_GAUSS15
gauss21  :: IntegrationRule
gauss21  = IntegrationRule GSL_INTEG_GAUSS21
.
.
.
```

The variables cannot be modified and are essentially constant flags. Since we did not export the IntegrationRule constructor in the module declaration, but only the gauss flags, it is impossible for a user to even construct an invalid value. One thing less to worry about!

Haskell Function Target

We can now write down the signature of the function we desire:

```
qag :: IntegrationRule                 -- Algorithm type
    -> Int                             -- Step limit
    -> Double                          -- Absolute tolerance
    -> Double                          -- Relative tolerance
    -> (Double -> Double)              -- Function to integrate
    -> Double                          -- Integration interval start
    -> Double                          -- Integration interval end
    -> Either String (Double, Double)  -- Result and (absolute) error estimate
```

Note how the order of arguments is different from the C version: indeed, since C does not have the possibility of partial application, the ordering criteria are different than in Haskell.

As in the previous example, we indicate errors with a Either String (Double, Double) result.

Passing Haskell Functions to the C Algorithm

```
type CFunction = CDouble -> Ptr () -> CDouble

data GslFunction = GslFunction (FunPtr CFunction) (Ptr ())
instance Storable GslFunction where
    sizeOf    _ = (#size gsl_function)
    alignment _ = alignment (undefined :: Ptr ())
    peek ptr = do
        function <- (#peek gsl_function, function) ptr
        return $ GslFunction function nullPtr
    poke ptr (GslFunction fun nullPtr) = do
        (#poke gsl_function, function) ptr fun

makeCfunction :: (Double -> Double) -> (CDouble -> Ptr () -> CDouble)
makeCfunction f = \x voidpointer -> realToFrac $ f (realToFrac x)

foreign import ccall "wrapper"
    makeFunPtr :: CFunction -> IO (FunPtr CFunction)
```

We define a shorthand type, CFunction, for readability. Note that the void pointer has been translated to a Ptr (), since we have no intention of using it. Then it is the turn of the gsl_function structure: no surprises here. Note that the void pointer is always assumed to be null, both in peek and in poke, and is never really read nor written.

To make a Haskell Double -> Double function available to the C algorithm, we make two steps: first, we re-organise the arguments using a λ function in makeCfunction; then, in makeFunPtr, we take the function with reordered arguments and produce a function pointer that we can pass on to poke, so we can construct the GslFunction data structure.

Handling Unknown Structures

```
data Workspace
foreign import ccall unsafe "gsl/gsl_integration.h
 gsl_integration_workspace_alloc"
    c_qag_alloc :: CSize -> IO (Ptr Workspace)
foreign import ccall unsafe "gsl/gsl_integration.h
 gsl_integration_workspace_free"
    c_qag_free  :: Ptr Workspace -> IO ()

foreign import ccall safe "gsl/gsl_integration.h gsl_integration_qag"
    c_qag :: Ptr GslFunction -- Allocated GSL function structure
          -> CDouble -- Start interval
          -> CDouble -- End interval
          -> CDouble -- Absolute tolerance
          -> CDouble -- Relative tolerance
          -> CSize   -- Maximum number of subintervals
          -> CInt    -- Type of Gauss-Kronrod rule
          -> Ptr Workspace -- GSL integration workspace
          -> Ptr CDouble -- Result
          -> Ptr CDouble -- Computation error
          -> IO CInt -- Exit code
```

The reason we imported the EmptyDataDecls pragma is this: we are declaring the data structure Workspace without providing any constructor. This is a way to make sure it will always be handled as a pointer, and never actually instantiated.

Otherwise, we normally import the allocating and deallocating routines. We can now import the integration function, since we have all the required pieces (GslFunction and Workspace).

The Complete Function

It is now possible to implement a function with the same functionality as the GSL's QAG algorithm.

```
qag gauss steps abstol reltol f a b = unsafePerformIO $ do
    c_deactivate_gsl_error_handler
    workspacePtr <- c_qag_alloc (fromIntegral steps)
    if workspacePtr == nullPtr
        then
            return $ Left "GSL could not allocate workspace"
        else do
            fPtr <- makeFunPtr $ makeCfunction f
            alloca $ \gsl_f -> do
                poke gsl_f (GslFunction fPtr nullPtr)
```

```
alloca $ \resultPtr -> do
    alloca $ \errorPtr -> do
        status <- c_qag gsl_f
                        (realToFrac a)
                        (realToFrac b)
                        (realToFrac abstol)
                        (realToFrac reltol)
                        (fromIntegral steps)
                        (rule gauss)
                        workspacePtr
                        resultPtr
                        errorPtr
        c_qag_free workspacePtr
        freeHaskellFunPtr fPtr
        if status /= 0
            then do
                c_errormsg <- c_error_string status
                errormsg   <- peekCString c_errormsg
                return $ Left errormsg
            else do
                c_result <- peek resultPtr
                c_error  <- peek  errorPtr
                let result = realToFrac c_result
                let error  = realToFrac c_error
                return $ Right (result, error)
```

First and foremost, we deactivate the GSL error handler, that would crash the program instead of letting us report the error.

We then proceed to allocate the workspace; notice that, if the returned pointer is null, there was an error (typically, too large size) that has to be reported.

If the workspace was allocated correctly, we convert the given function to a function pointer and allocate the `GslFunction` struct, in which we place the function pointer. Allocating memory for the result and its error margin is the last thing before calling the main routine.

After calling, we have to do some housekeeping and free the memory allocated by the workspace and the function pointer. Note that it would be possible to skip the bookkeeping using `ForeignPtr`, but the work required to get it to work is more than the effort to remember one line of cleanup.

We then proceed to check the return value and return the result, as was done for the Bessel function.

82.1.7 Self-Deallocating Pointers

In the previous example, we manually handled the deallocation of the GSL integration workspace, a data structure we know nothing about, by calling its C deallocation function. It happens that the same workspace is used in several integration routines, which we may want to import in Haskell.

Instead of replicating the same allocation/deallocation code each time, which could lead to memory leaks when someone forgets the deallocation part, we can provide a sort of ”smart pointer”, which will deallocate the memory when it is not needed any more. This is called `ForeignPtr` (do not confuse with `Foreign.Ptr`: this one's qualified name is actually `Foreign.ForeignPtr`!). The function handling the deallocation is called the *finalizer*.

In this section we will write a simple module to allocate GSL workspaces and provide them as appropriately configured `ForeignPtr`s, so that users do not have to worry about deallocation.

The module, written in file `GSLWorkspace.hs`, is as follows:

```
{-# LANGUAGE ForeignFunctionInterface, EmptyDataDecls #-}

module GSLWorkSpace (Workspace, createWorkspace) where

import Foreign.C.Types
import Foreign.Ptr
import Foreign.ForeignPtr

data Workspace
foreign import ccall unsafe "gsl/gsl_integration.h
 gsl_integration_workspace_alloc"
    c_ws_alloc :: CSize -> IO (Ptr Workspace)
foreign import ccall unsafe "gsl/gsl_integration.h
 &gsl_integration_workspace_free"
    c_ws_free  :: FunPtr( Ptr Workspace -> IO () )

createWorkspace :: CSize -> IO (Maybe (ForeignPtr Workspace) )
createWorkspace size = do
    ptr <- c_ws_alloc size
    if ptr /= nullPtr
        then do
           foreignPtr <- newForeignPtr c_ws_free ptr
           return $ Just foreignPtr
        else
           return Nothing
```

We first declare our empty data structure `Workspace`, just like we did in the previous section.

The `gsl_integration_workspace_alloc` and `gsl_integration_workspace_free` functions will no longer be needed in any other file: here, note that the deallocation function is called with an ampersand ("&"), because we do not actually want the function, but rather a *pointer* to it to set as a finalizer.

The workspace creation function returns a IO (Maybe) value, because there is still the possibility that allocation is unsuccessful and the null pointer is returned. The GSL does not specify what happens if the deallocation function is called on the null pointer, so for safety we do not set a finalizer in that case and return `IO Nothing`; the user code will then have to check for "`Just`-ness" of the returned value.

If the pointer produced by the allocation function is non-null, we build a foreign pointer with the deallocation function, inject into the `Maybe` and then the `IO` monad. That's it, the foreign pointer is ready for use!

⚠ Warning

This function requires object code to be compiled, so if you load this module with GHCI (which is an interpreter) you must indicate it:

```
$ ghci GSLWorkSpace.hs -fobject-code
```

Or, from within GHCI:

```
> :set -fobject-code
> :load GSLWorkSpace.hs
```

The qag.hsc file must now be modified to use the new module; the parts that change are:

```
{-# LANGUAGE ForeignFunctionInterface #-}

-- [...]

import GSLWorkSpace

import Data.Maybe(isNothing, fromJust)

-- [...]

qag gauss steps abstol reltol f a b = unsafePerformIO $ do
    c_deactivate_gsl_error_handler
    ws <- createWorkspace (fromIntegral steps)
    if isNothing ws
        then
            return $ Left "GSL could not allocate workspace"
        else do
            withForeignPtr (fromJust ws) $ \workspacePtr -> do

-- [...]
```

Obviously, we do not need the EmptyDataDecls extension here any more; instead we import the GSLWorkSpace module, and also a couple of nice-to-have functions from Data.Maybe. We also remove the foreign declarations of the workspace allocation and deallocation functions.

The most important difference is in the main function, where we (try to) allocate a workspace ws, test for its Justness, and if everything is fine we use the withForeignPtr function to extract the workspace pointer. Everything else is the same.

82.2 Calling Haskell from C

Sometimes it is also convenient to call Haskell from C, in order to take advantage of some of Haskell's features which are tedious to implement in C, such as lazy evaluation.

We will consider a typical Haskell example, Fibonacci numbers. These are produced in an elegant, haskellian one-liner as:

```
fibonacci = 0 : 1 : zipWith (+) fibonacci (tail fibonacci)
```

Our task is to export the ability to calculate Fibonacci numbers from Haskell to C. However, in Haskell, we typically use the Integer type, which is unbounded: this cannot be exported to C, since there is no such corresponding type. To provide a larger range of outputs, we specify that the C function shall output, whenever the result is beyond the bounds of its integer type, an approximation in floating-point. If the result is also beyond the range of floating-point, the computation will fail. The status of the result (whether it can be represented as a C integer, a floating-point type or not at all) is signalled by the status integer returned by the function. Its desired signature is therefore:

```
int fib( int index, unsigned long long* result, double* approx )
```

82.2.1 Haskell Source

The Haskell source code for file `fibonacci.hs` is:

```
{-# LANGUAGE ForeignFunctionInterface #-}

module Fibonacci where

import Foreign
import Foreign.C.Types

fibonacci :: (Integral a) => [a]
fibonacci = 0 : 1 : zipWith (+) fibonacci (tail fibonacci)

foreign export ccall fibonacci_c :: CInt -> Ptr CULLong -> Ptr CDouble -> IO
 CInt
fibonacci_c :: CInt -> Ptr CULLong -> Ptr CDouble -> IO CInt
fibonacci_c n intPtr dblPtr
    | badInt && badDouble = return 2
    | badInt              = do
        poke dblPtr dbl_result
        return 1
    | otherwise           = do
        poke intPtr (fromIntegral result)
        poke dblPtr dbl_result
        return 0
    where
    result     = fibonacci !! (fromIntegral n)
    dbl_result = realToFrac result
    badInt     = result > toInteger (maxBound :: CULLong)
    badDouble  = isInfinite dbl_result
```

When exporting, we need to wrap our functions in a module (it is a good habit anyway). We have already seen the Fibonacci infinite list, so let's focus on the exported function: it takes an argument, two pointers to the target `unsigned long long` and `double`, and returns the status in the `IO` monad (since writing on pointers is a side effect).

The function is implemented with input guards, defined in the `where` clause at the bottom. A successful computation will return 0, a partially successful 1 (in which we still can use the floating-point value as an approximation), and a completely unsuccessful one will return 2.

Note that the function does not call `alloca`, since the pointers are assumed to have been already allocated by the calling C function.

The Haskell code can then be compiled with GHC:

```
ghc -c fibonacci.hs
```

82.2.2 C Source

The compilation of `fibonacci.hs` has spawned several files, among which `fibonacci_stub.h`, which we include in our C code in file `fib.c`:

```
#include <stdio.h>
#include <stdlib.h>
#include "fibonacci_stub.h"

int main(int argc, char *argv[]) {
    if (argc < 2) {
        printf("Usage: %s <number>\n", argv[0]);
        return 2;
    }

    hs_init(&argc, &argv);

    const int arg = atoi(argv[1]);
    unsigned long long res;
    double approx;
    const int status = fibonacci_c(arg, &res, &approx);

    hs_exit();
    switch (status) {
    case 0:
        printf("F_%d: %llu\n", arg, res);
        break;
    case 1:
        printf("Error: result is out of bounds\n");
        printf("Floating-point approximation: %e\n", approx);
        break;
    case 2:
        printf("Error: result is out of bounds\n");
        printf("Floating-point approximation is infinite\n");
        break;
    default:
        printf("Unknown error: %d\n", status);
    }

    return status;
}
```

The notable thing is that we need to initialise the Haskell environment with `hs_init`, which we call passing it the command-line arguments of main; we also have to shut Haskell down with `hs_exit()` when we are done. The rest is fairly standard C code for allocation and error handling.

Note that you have to compile the C code *with GHC*, not your C compiler!

```
ghc -no-hs-main fib.c fibonacci.o fibonacci_stub.o -o fib
```

You can then proceed to test the algorithm:

```
./fib 42
F_42: 267914296
$ ./fib 666
Error: result is out of bounds
Floating-point approximation: 6.859357e+138
$ ./fib 1492
```

```
Error: result is out of bounds
Floating-point approximation is infinite
./fib -1
fib: Prelude.(!!): negative index
```

83 Generic Programming : Scrap your boilerplate

The "Scrap your boilerplate" approach, "described" in `http://www.cs.vu.nl/boilerplate/`, is a way to allow your data structures to be traversed by so-called "generic" functions: that is, functions that abstract over the specific data constructors being created or modified, while allowing for the addition of cases for specific types.

For instance if you want to serialize all the structures in your code, but you want to write only one serialization function that operates over any instance of the Data.Data.Data class (which can be derived with -XDeriveDataTypeable).

83.1 Serialization Example

The goal is to convert all our data into a format below:

```
data Tag = Con String | Val String
```

83.2 Comparing Haskell ASTs

The haskell-src-exts package[1] parses Haskell into a quite complicated syntax tree. Let's say we want to check if two source files that are nearly identical are equivalent.

To start:

```
import System.Environment
import Language.Haskell.Exts

main = do
   -- parse the filenames given by the first two command line arguments,
   -- proper error handling is left as an exercise
   [ParseOk moduleA, ParseOk moduleB] <- mapM parseFile . take 2 =<< getArgs

   putStrLn $ if moduleA == moduleB
        then "Your modules are equal"
        else "Your modules differ"
```

From a bit of testing, it will be apparent that identical files with different names will not be equal to (==). However, to correct the fact, without resorting to lots of boilerplate, we can use generic programming:

[1] http://hackage.haskell.org/package/haskell-src-exts

83.3 TODO

describe using Data.Generics.Twins.gzip*? to write a function to find where there are differences?

Or use it to write a variant of geq that ignores the specific cases that are unimportant (the SrcLoc elements) (i.e. syb doesn't allow generic extension... contrast it with other libraries?).

Or just explain this hack (which worked well enough) to run before (==), or geq::

```
everyWhere (mkT $ \ _ -> SrcLoc "" 0 0) :: Data a => a -> a
```

Or can we develop this into writing something better than sim_mira (for hs code), found here: `http://dickgrune.com/Programs/similarity_tester/`

84 Specialised Tasks

85 Graphical user interfaces (GUI)

Haskell has at least four toolkits for programming a graphical interface:

- wxHaskell[1] - provides a Haskell interface to the cross-platform wxWidgets toolkit which supports Windows, OS X, and Gtk+ on GNU/Linux, among others.
- Gtk2Hs[2] - provides a Haskell interface to the GTK+ library
- hoc[3] (documentation at sourceforge[4]) - provides a Haskell to Objective-C binding which allows users to access to the Cocoa library on MacOS X
- qtHaskell[5] - provides a set of Haskell bindings for the Qt Widget Library

In this tutorial, we will focus on the wxHaskell toolkit.

85.1 Getting and running wxHaskell

To install wxHaskell, look for your version of instructions at: GNU/Linux[6] Mac[7] Windows[8]

or the wxHaskell download page[9] and follow the installation instructions provided on the wxHaskell download page. Don't forget to register wxHaskell with GHC, or else it won't run (automatically registered with Cabal). To compile source.hs (which happens to use wxHaskell code), open a command line and type:

```
ghc -package wx source.hs -o bin
```

Code for GHCi is similar:

```
ghci -package wx
```

You can then load the files from within the GHCi interface. To test if everything works, go to $wxHaskellDir/samples/wx ($wxHaskellDir is the directory where you installed it) and load (or compile) HelloWorld.hs. It should show a window with title ”Hello World!”,

[1] https://en.wikibooks.org/wiki/%3Aw%3AWxHaskell
[2] http://www.haskell.org/haskellwiki/Gtk2Hs
[3] http://code.google.com/p/hoc/
[4] http://hoc.sourceforge.net/
[5] http://qthaskell.berlios.de/
[6] http://www.haskell.org/haskellwiki/WxHaskell/Linux
[7] http://www.haskell.org/haskellwiki/WxHaskell/Mac
[8] http://www.haskell.org/haskellwiki/WxHaskell/Windows
[9] http://wxhaskell.sourceforge.net/download.html

a menu bar with File and About, and a status bar at the bottom, that says ”Welcome to wxHaskell”.

If it doesn’t work, you might try to copy the contents of the $wxHaskellDir/lib directory to the ghc install directory.

85.1.1 Shortcut for Debian and Ubuntu

If your operating system is Debian or Ubuntu, you can simply run these commands from the terminal:

```
sudo apt-get install g++
sudo apt-get install libglu-dev
sudo apt-get install libwxgtk2.8-dev
```

85.2 Hello World

Here’s the basic Haskell ”Hello World” program:

```
module Main where

main :: IO ()
main = putStr "Hello World!"
```

It will compile just fine, but how do we actually do GUI work with this? First, you must import the wxHaskell library `Graphics.UI.WX`. `Graphics.UI.WXCore` has some more stuff, but we won’t need that now.

To start a GUI, use `start gui`. In this case, `gui` is the name of a function which we’ll use to build the interface. It must have an IO type. Let’s see what we have:

```
module Main where

import Graphics.UI.WX

main :: IO ()
main = start gui

gui :: IO ()
gui = do
  --GUI stuff
```

To make a frame, we use `frame` which has the type `[Prop (Frame ())] -> IO (Frame ())`. It takes a list of ”frame properties” and returns the corresponding frame. We’ll look deeper into properties later, but a property is typically a combination of an attribute and a value. What we’re interested in now is the title. This is in the `text` attribute and has type `(Textual w) => Attr w String`. The most important thing here, is that it’s a `String` attribute. Here’s how we code it:

```
gui :: IO ()
gui = do
  frame [text := "Hello World!"]
```

The operator `(:=)` takes an attribute and a value and combines both into a property. Note that `frame` returns an `IO (Frame ())`. You can change the type of `gui` to `IO (Frame ())`, but it might be better just to add `return ()`. Now we have our own GUI consisting of a frame with title ”Hello World!”. Its source:

```
module Main where

import Graphics.UI.WX

main :: IO ()
main = start gui

gui :: IO ()
gui = do
  frame [text := "Hello World!"]
  return ()
```

The result should look like the screenshot. (It might look slightly different on Linux or MacOS X, on which wxhaskell also runs)

85.3 Controls

> **ℹ Information**
>
> From here on, its good practice to keep a browser window or tab open with the wxHaskell documentation[10]. It's also available in $wxHaskellDir/doc/index.html.

85.3.1 A text label

A simple frame doesn't do much. In this section, we're going to add some more elements. Let's start simple with a label. wxHaskell has a `label`, but that's a layout thing. We won't be doing layout until next section. What we're looking for is a `staticText`. It's in `Graphics.UI.WX.Controls`. The `staticText` function takes a `Window` as argument along with a list of properties. Do we have a window? Yup! Look at `Graphics.UI.WX.Frame`. There, we see that a `Frame` is merely a type-synonym of a special sort of window. We'll change the code in `gui` so it looks like this:

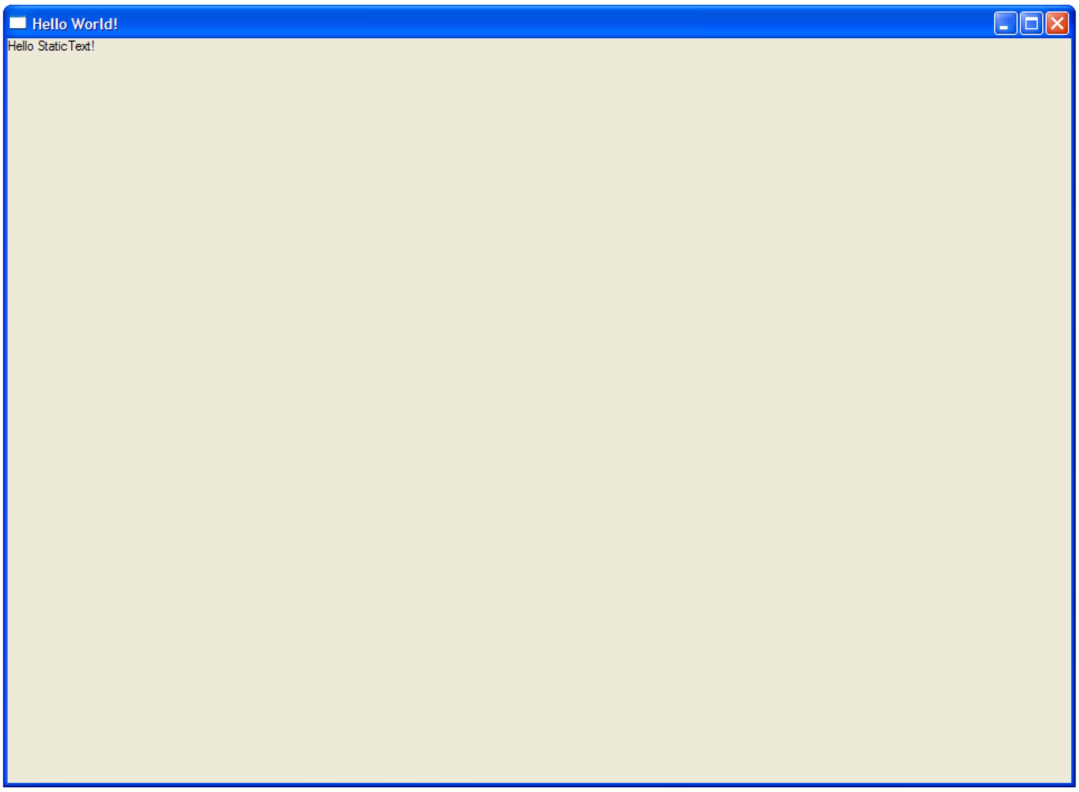


Figure 37 Hello StaticText! (winXP)

```
gui :: IO ()
gui = do
  f <- frame [text := "Hello World!"]
  staticText f [text := "Hello StaticText!"]
  return ()
```

Again, `text` is an attribute of a `staticText` object, so this works. Try it!

85.3.2 A button

Now for a little more interaction. A button. We're not going to add functionality to it until the section about events, but already something visible will happen when you click on it.

A `button` is a control, just like `staticText`. Look it up in `Graphics.UI.WX.Controls`.

Again, we need a window and a list of properties. We'll use the frame again. `text` is also an attribute of a button:

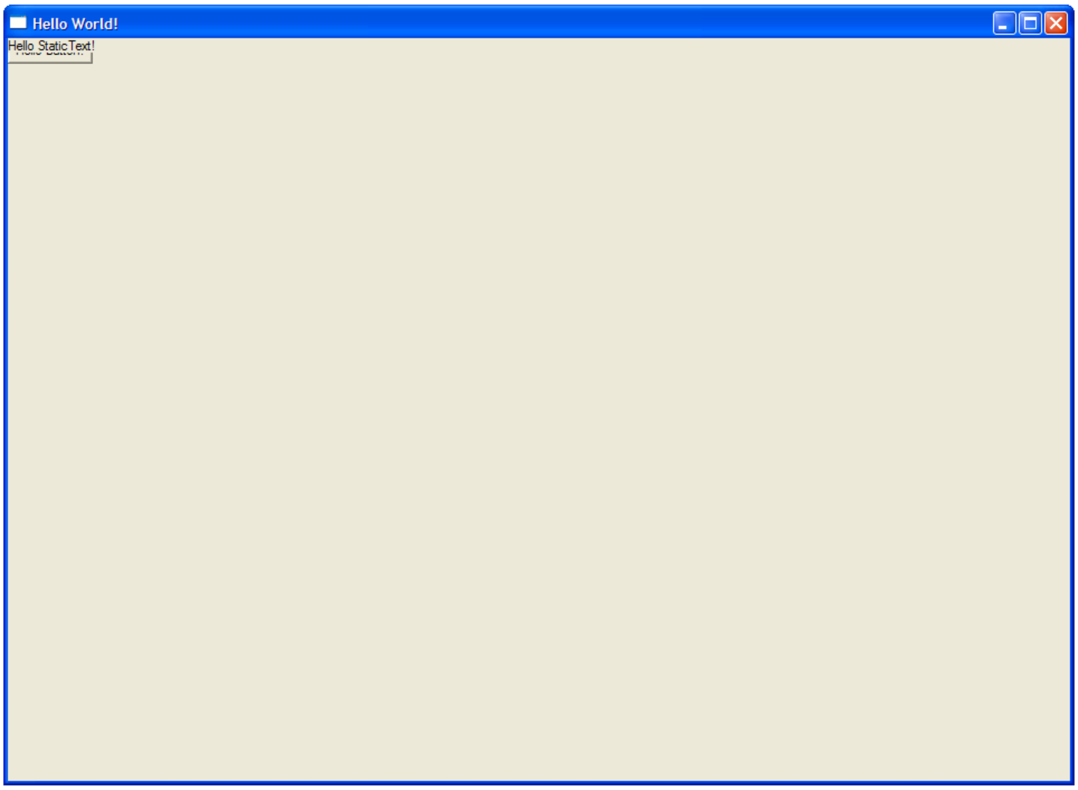


Figure 38 Overlapping button and StaticText (winXP)

```
gui :: IO ()
gui = do
  f <- frame [text := "Hello World!"]
  staticText f [text := "Hello StaticText!"]
  button f [text := "Hello Button!"]
  return ()
```

Load it into GHCi (or compile it with GHC) and... hey!? What's that? The button's been covered up by the label! We're going to fix that next.

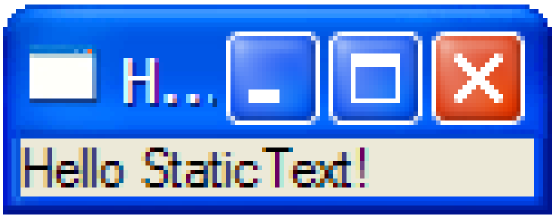
85.4 Layout

The reason that the label and the button overlap, is that we haven't set a *layout* for our frame yet. Layouts are created using the functions found in the documentation of `Graphics.UI.WXCore.Layout`. Note that you don't have to import `Graphics.UI.WXCore` to use layouts.

The documentation says we can turn a member of the widget class into a layout by using the `widget` function. Also, windows are a member of the widget class. But, wait a minute... we only have one window, and that's the frame! Nope... we have more, look at `Graphics.UI.WX.Controls` and click on any occurrence of the word *Control*. You'll be taken to `Graphics.UI.WXCore.WxcClassTypes`, and it is there we see that a Control is also a type synonym of a special type of window. We'll need to change the code a bit, but here it is.

```
gui :: IO ()
gui = do
  f <- frame [text := "Hello World!"]
  st <- staticText f [text := "Hello StaticText!"]
  b <- button f [text := "Hello Button!"]
  return ()
```

Now we can use `widget st` and `widget b` to create a layout of the staticText and the button. `layout` is an attribute of the frame, so we'll set it here:



Hello StaticText!

Figure 39 StaticText with layout (winXP)

```
gui :: IO ()
gui = do
  f <- frame [text := "Hello World!"]
  st <- staticText f [text := "Hello StaticText!"]
  b <- button f [text := "Hello Button!"]
  set f [layout := widget st]
  return ()
```

The `set` function will be covered in the section below about attributes. Try the code, what's wrong? This only displays the staticText, not the button. We need a way to combine the two. We will use *layout combinators* for that. `row` and `column` look nice. They take an integer and a list of layouts. We can easily make a list of layouts of the button and the staticText. The integer is the spacing between the elements of the list. Let's try something:

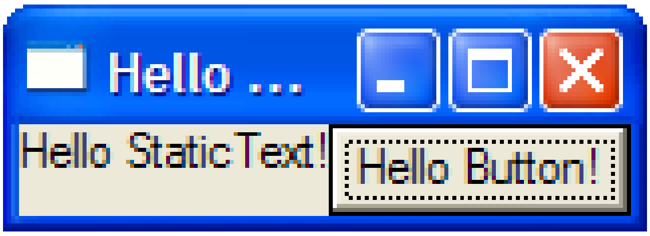


Figure 40 A row layout (winXP)

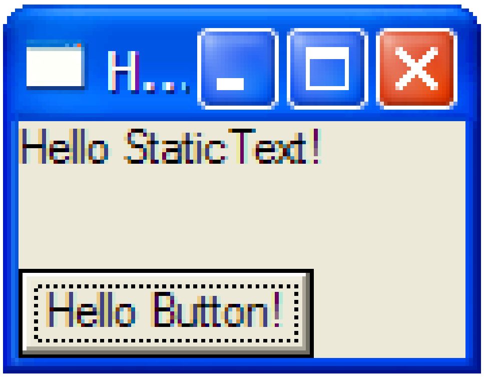


Figure 41 Column layout with a spacing of 25 (winXP)

```
gui :: IO ()
gui = do
  f <- frame [text := "Hello World!"]
  st <- staticText f [text := "Hello StaticText!"]
  b <- button f [text := "Hello Button!"]
  set f [layout :=
          row 0 [widget st, widget b]
        ]
  return ()
```

Play around with the integer and see what happens. Also, change `row` into `column`. Try to change the order of the elements in the list to get a feeling of how it works. For fun, try to add `widget b` several more times in the list. What happens?

Here are a few exercises to spark your imagination. Remember to use the documentation!

Exercises:

1. Add a checkbox control. It doesn't have to do anything yet, just make sure it appears next to the staticText and the button when using row-layout, or below them when using column layout. `text` is also an attribute of the checkbox.
2. Notice that `row` and `column` take a list of *layouts*, and also generates a layout itself. Use this fact to make your checkbox appear on the left of the staticText and the button, with the staticText and the button in a column.
3. Can you figure out how the radiobox control works? Take the layout of the previous exercise and add a radiobox with two (or more) options below the checkbox, staticText and button. Use the documentation!
4. Use the `boxed` combinator to create a nice looking border around the four controls, and another one around the staticText and the button. (*Note: the `boxed` combinator might not be working on MacOS X - you might get widgets that can't be interacted with. This is likely just a bug in wxhaskell.*)

After having completed the exercises, the end result should look like this:

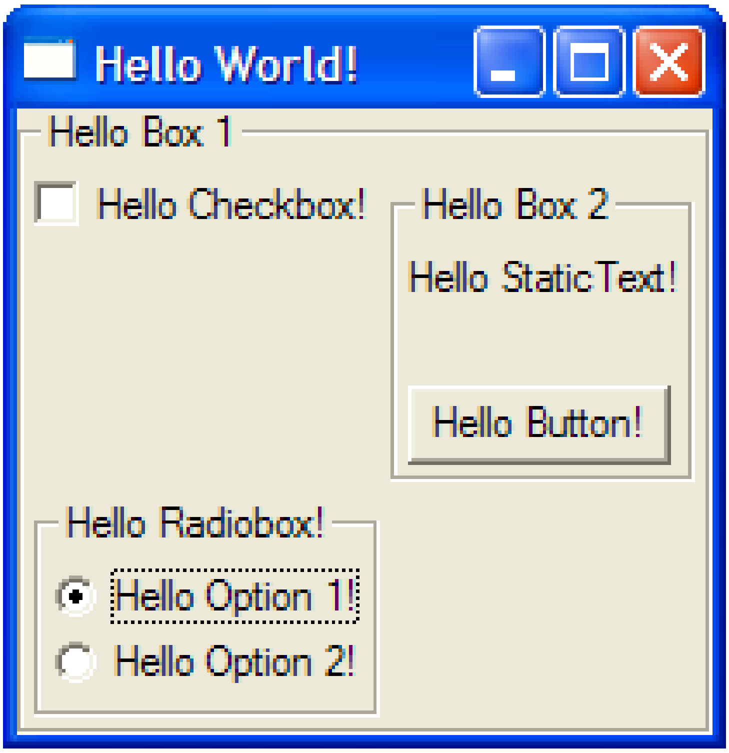


Figure 42 Answer to exercises

You could have used different spacing for `row` and `column` or have the options of the radiobox displayed horizontally.

85.5 Attributes

After all this, you might be wondering: ”Where did that `set` function suddenly come from?” and ”How would *I* know if `text` is an attribute of something?”. Both answers lie in the attribute system of wxHaskell.

85.5.1 Setting and modifying attributes

In a wxHaskell program, you can set the properties of the widgets in two ways:

1. during creation: `f <- frame [ text := "Hello World!" ]`
2. using the `set` function: `set f [ layout := widget st ]`

The `set` function takes two arguments: something of type `w` along with properties of `w`. In wxHaskell, these will be the widgets and the properties of these widgets. Some properties can only be set during creation, such as the `alignment` of a `textEntry`, but you can set most others in any IO-function in your program — as long as you have a reference to it (the `f` in `set f [`*`stuff`*`]`).

Apart from setting properties, you can also get them. This is done with the `get` function. Here's a silly example:

```
gui :: IO ()
gui = do
  f <- frame [ text := "Hello World!" ]
  st <- staticText f []
  ftext <- get f text
  set st [ text := ftext]
  set f [ text := ftext ++ " And hello again!" ]
```

Look at the type signature of `get`. It's `w -> Attr w a -> IO a`. `text` is a `String` attribute, so we have an `IO String` which we can bind to `ftext`. The last line edits the text of the frame. Yep, destructive updates are possible in wxHaskell. We can overwrite the properties using `(:=)` anytime with `set`. This inspires us to write a modify function:

```
modify :: w -> Attr w a -> (a -> a) -> IO ()
modify w attr f = do
  val <- get w attr
  set w [ attr := f val ]
```

First it gets the value, then it sets it again after applying the function. Surely we're not the first one to think of that...

Look at this operator: `(:~)`. You can use it in `set` because it takes an attribute and a function. The result is a property in which the original value is modified by the function. That means we can write:

```
gui :: IO ()
gui = do
  f <- frame [ text := "Hello World!" ]
  st <- staticText f []
  ftext <- get f text
  set st [ text := ftext ]
  set f [ text :~ ++ " And hello again!" ]
```

This is a great place to use anonymous functions with the lambda-notation.

There are two more operators we can use to set or modify properties: `(::=)` and `(::~)`. They do almost the same as `(:=)` and `(:~)` except a function of type `w -> orig` is expected, where `w` is the widget type, and `orig` is the original ”value” type (`a` in case of `(:=)` and `a -> a` in case of `(:~)`). We won't be using them now, as we've only encountered attributes of non-IO types, and the widget needed in the function is generally only useful in IO-blocks.

85.5.2 How to find attributes

Now the second question. Where do we go to determine that `text` is an attribute of all those things? Go to the documentation...

Let's see what attributes a button has: Go to `Graphics.UI.WX.Controls`[11]. Click the link that says "Button"[12]. You'll see that a `Button` is a type synonym of a special kind of `Control`, and a list of functions that can be used to create a button. After each function, there's a list of "Instances". For the normal `button` function, we see *Commanding -- Textual, Literate, Dimensions, Colored, Visible, Child, Able, Tipped, Identity, Styled, Reactive, Paint*. That's the list of classes of which a button is an instance. Read through the ../Classes and types/[13] chapter. It means that there are some class-specific functions available for the button. `Textual`, for example, adds the `text` and `appendText` functions. If a widget is an instance of the `Textual` class, it means that it has a `text` attribute!

Note that while `StaticText` hasn't got a list of instances, it's still a `Control`, and that's a synonym for some kind of `Window`. When looking at the `Textual` class, it says that `Window` is an instance of it. That's an error on the side of the documentation!

Let's take a look at the attributes of a frame. They can be found in `Graphics.UI.WX.Frame`. Another error in the documentation here: It says `Frame` instantiates `HasImage`. This was true in an older version of wxHaskell. It should say `Pictured`. Apart from that, we have `Form`, `Textual`, `Dimensions`, `Colored`, `Able` and a few more. We're already seen `Textual` and `Form`. Anything that is an instance of `Form` has a `layout` attribute.

`Dimensions` adds (among others) the `clientSize` attribute. It's an attribute of the `Size` type, which can be made with `sz`. Please note that the `layout` attribute can also change the size. If you want to use `clientSize` you should set it after the `layout`.

`Colored` adds the `color` and `bgcolor` attributes.

`Able` adds the Boolean `enabled` attribute. This can be used to enable or disable certain form elements, which is often displayed as a greyed-out option.

There are lots of other attributes, read through the documentation for each class.

85.6 Events

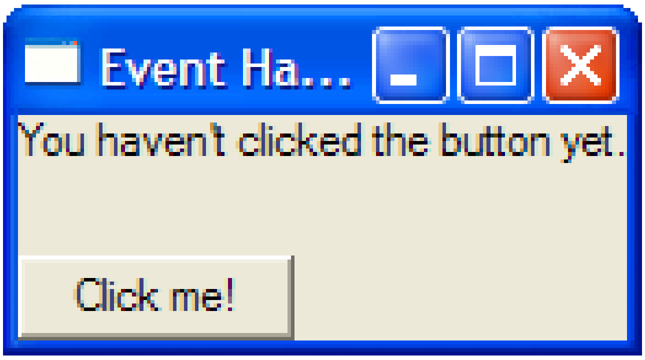
There are a few classes that deserve special attention. They are the `Reactive` class and the `Commanding` class. As you can see in the documentation of these classes, they don't add attributes (of the form `Attr w a`), but *events*. The `Commanding` class adds the `command` event. We'll use a button to demonstrate event handling.

Here's a simple GUI with a button and a staticText:

[11] http://hackage.haskell.org/packages/archive/wx/0.10.2/doc/html/Graphics-UI-WX-Controls.html

[12] http://hackage.haskell.org/packages/archive/wx/0.10.2/doc/html/Graphics-UI-WX-Controls.html#4

[13] Chapter 26 on page 155



You haven't clicked the button yet.

Figure 43 Before (winXP)

```
gui :: IO ()
gui = do
  f <- frame [ text := "Event Handling" ]
  st <- staticText f [ text := "You haven\'t clicked the button yet." ]
  b <- button f [ text := "Click me!" ]
  set f [ layout := column 25 [ widget st, widget b ] ]
```

We want to change the staticText when you press the button. We'll need the `on` function:

```
   b <- button f [ text := "Click me!"
                 , on command := --stuff
                 ]
```

The type of `on`: `Event w a -> Attr w a`. `command` is of type `Event w (IO ())`, so we need an IO-function. This function is called the *Event handler*. Here's what we get:

```
gui :: IO ()
gui = do
  f <- frame [ text := "Event Handling" ]
  st <- staticText f [ text := "You haven\'t clicked the button yet." ]
  b <- button f [ text := "Click me!"
                , on command := set st [ text := "You have clicked the button!"
 ]
                ]
  set f [ layout := column 25 [ widget st, widget b ] ]
```

Insert text about event filters here

86 Databases

86.1 Introduction

Haskell's most popular database module is HDBC[1]. HDBC provides an abstraction layer between Haskell programs and SQL relational databases. This lets you write database code once, in Haskell, and have it work with a number of backend SQL databases.

HDBC is modeled loosely on Perl's DBI interface[2], though it has also been influenced by Python's DB-API v2, JDBC in Java, and HSQL in Haskell. Like how DBI requires DBD in Perl, HDBC requires a driver module beneath it to work.

These HDBC backend drivers exist: PostgreSQL, SQLite, and ODBC (for Windows and Unix/Linux/Mac). MySQL is the most popular open-sourced databases, and there are two drivers for MySQL: HDBC-mysql[3] (native) and HDBC-odbc[4] (ODBC). MySQL users can use the ODBC driver on any MySQL-supported platform, including Linux.

An advantage of using ODBC is that the syntax of the SQL statement is insulated from the different kinds of database engines. This increases the portability of the application should you have to move from one database to another. The same argument for preferring ODBC applies for other commercial databases (such as Oracle and DB2).

86.2 Installation

86.2.1 PostgreSQL or SQLite

See the HDBC FAQ[5] for more information.

86.2.2 Native MySQL

The native ODBC-mysql library requires the C MySQL client library to be present.

You may need to wrap your database accesses[6] to prevent runtime errors.

[1] https://github.com/hdbc/hdbc/wiki

[2] http://search.cpan.org/~timb/DBI/DBI.pm

[3] http://hackage.haskell.org/package/HDBC-mysql

[4] http://hackage.haskell.org/package/HDBC-odbc

[5] https://github.com/hdbc/hdbc/wiki/FrequentlyAskedQuestions

[6] http://www.serpentine.com/blog/2010/09/04/dealing-with-fragile-c-libraries-e-g-mysql-from-haskell/

86.2.3 ODBC/MySQL

Making HDBC work with MySQL via ODBC is somewhat involved, especially if you do not have root privileges.

- If your platform doesn't already provide an ODBC library (and most do), install Unix-ODBC. See here[7] for more information.
- Install MySQL-ODBC Connector. See here[8] for more information.
- Install Database.HDBC module
- Install Database.HDBC.ODBC module
- Add the mysql driver to odbcinst.ini file (under $ODBC_HOME/etc/) and your data source in $HOME/.odbc.ini.
- Create a test program

Since the ODBC driver is installed using shared library by default, you will need the following env:

```
export LD_LIBRARY_PATH=$ODBC_HOME/lib
```

If you do not like adding an additional env variables, you should try to compile ODBC with static library option enabled.

The next task is to write a simple test program that connects to the database and print the names of all your tables, as shown below.

You may need to wrap your database accesses[9] in order to prevent runtime errors.

```
module Main where
import Database.HDBC.ODBC
import Database.HDBC
main =
  do c  <- connectODBC "DSN=PSPDSN"
     xs <- getTables c
     putStr $ "tables "++(foldr jn "." xs)++"\n"
  where jn a b = a++" "++b
```

86.3 General Workflow

86.3.1 Connect and Disconnect

The first step of any database operation is to connect to the target database. This is done via the driver-specific connect API, which has the type of:

```
String -> IO Connection
```

Given a connect string, the connect API will return `Connection` and put you in the IO monad.

[7] http://sourceforge.net/projects/unixodbc/

[8] http://dev.mysql.com/downloads/connector/odbc/

[9] http://www.serpentine.com/blog/2010/09/04/dealing-with-fragile-c-libraries-e-g-mysql-from-haskell/

Although most programs will garbage-collect your connections when they are out of scope or when the program ends, it is a good practice to disconnect from the database explicitly.

```
conn->Disconnect
```

86.3.2 Running Queries

Running a query generally involves the following steps:

- Prepare a statement
- Execute a statement with bind variables
- Fetch the result set (if any)
- Finish the statement

HDBC provides two ways for bind variables and returning result set: `[ SqlValue ]` and `[ Maybe String ]`. You need to use the functions with `s` prefix when using `[ Maybe String ]`, instead of `[ SqlValue ]`. `[ SqlValue ]` allows you to use strongly typed data if type safety is very important in your application; otherwise, `[ Maybe String ]` is more handy when dealing with lots of database queries. When you use `[ Maybe String ]`, you assume the database driver will perform automatic data conversion. Be aware there is a performance price for this convenience.

Sometimes, when the query is simple, there are simplified APIs that wrap multiple steps into one. For example, **Run** and **sRun** are wrappers of ”prepare and execute”. **quickQuery** is a wrapper of ”prepare, execute, and fetch all rows”.

86.4 Running SQL Statements

86.4.1 Select

86.4.2 Insert

86.4.3 Update

86.4.4 Delete

86.5 Transaction

Database transaction is controlled by `commit` and `rollback`. However, be aware some databases (such as mysql) do not support transaction. Therefore, every query is in its atomic transaction.

HDBC provides `withTransaction` to allow you automate the transaction control over a group of queries.

86.6 Calling Procedure

87 Web programming

An example web application, using the HAppS framework, is hpaste[1], the Haskell paste bin. Built around the core Haskell web framework, HAppS, with HaXmL for page generation, and binary/zlib for state serialisation.

The HTTP and Browser modules[2] exist, and might be useful.

Category:Haskell/Not in book[3]

[1] http://hpaste.org

[2] http://homepages.paradise.net.nz/warrickg/haskell/http/

[3] https://en.wikibooks.org/wiki/Category%3AHaskell%2FNot%20in%20book

88 Working with XML

There are several Haskell libraries for XML work, and additional ones for HTML. For more web-specific work, you may want to refer to the Haskell/Web programming[1] chapter.

88.0.1 Libraries for parsing XML

- The Haskell XML Toolbox (hxt)[2] is a collection of tools for parsing XML, aiming at a more general approach than the other tools.
- HaXml[3] is a collection of utilities for parsing, filtering, transforming, and generating XML documents using Haskell.
- HXML[4] is a non-validating, lazy, space efficient parser that can work as a drop-in replacement for HaXml.
- xml-conduit[5] provides parsing and rendering functions for XML. For a tutorial see `http://www.yesodweb.com/book/xml`.

88.0.2 Libraries for generating XML

- HSXML represents XML documents as statically typesafe s-expressions.

88.0.3 Other options

- tagsoup[6] is a library for parsing unstructured HTML, i.e. it does not assume validity or even well-formedness of the data.

88.1 Getting acquainted with HXT

In the following, we are going to use the Haskell XML Toolbox for our examples. You should have a working installation of GHC[7], including GHCi, and you should have downloaded and installed HXT according to the instructions[8].

[1] Chapter 87 on page 573
[2] `http://www.fh-wedel.de/~si/HXmlToolbox/`
[3] `http://projects.haskell.org/HaXml/`
[4] `http://www.flightlab.com/~joe/hxml/`
[5] `https://hackage.haskell.org/package/xml-conduit`
[6] `http://www.cs.york.ac.uk/fp/darcs/tagsoup/tagsoup.htm`
[7] Chapter 2 on page 5
[8] `http://www.fh-wedel.de/~si/HXmlToolbox/#install`

With those in place, we are ready to start playing with HXT. Let’s bring the XML parser into scope, and parse a simple XML-formatted string:

```
 Prelude> :m + Text.XML.HXT.Parser.XmlParsec
 Prelude Text.XML.HXT.Parser.XmlParsec> xread ”<foo>abc<bar/>def</foo>”
[NTree (XTag (QN {namePrefix = ””, localPart = ”foo”, namespaceUri = ””}) [])
[NTree (XText ”abc”) [],NTree (XTag (QN {namePrefix = ””, localPart = ”bar”,
namespaceUri = ””}) []) [],NTree (XText ”def”) []]]
```

We see that HXT represents an XML document as a list of trees, where the nodes can be constructed as an XTag containing a list of subtrees, or an XText containing a string. With GHCi, we can explore this in more detail:

```
 Prelude> :m + Data.Tree.NTree.TypeDefs
 Prelude Text.XML.HXT.Parser.XmlParsec Text.XML.HXT.DOM> :i NTree
data NTree a = NTree a (NTrees a)
                        -- Defined in Data.Tree.NTree.TypeDefs
Prelude Text.XML.HXT.Parser.XmlParsec Text.XML.HXT.DOM> :i NTrees
type NTrees a = [NTree a]        -- Defined in Data.Tree.NTree.TypeDefs
```

As we can see, an NTree is a general tree structure where a node stores its children in a list, and some more browsing around will tell us that XML documents are trees over an XNode type, defined as:

```
data XNode
  = XText String
  | XCharRef Int
  | XEntityRef String
  | XCmt String
  | XCdata String
  | XPi QName XmlTrees
  | XTag QName XmlTrees
  | XDTD DTDElem Attributes
  | XAttr QName
  | XError Int String
```

Returning to our example, we notice that while HXT successfully parsed our input, one might desire a more lucid presentation for human consumption. Lucky for us, the DOM module supplies this. Notice that xread returns a list of trees, while the formatting function works on a single tree.

```
  Prelude Text.XML.HXT.Parser.XmlParsec> :m + Text.XML.HXT.DOM.FormatXmlTree
  Prelude Text.XML.HXT.Parser.XmlParsec Text.XML.HXT.DOM> putStrLn $ formatXmlTree $ head $ xread
”<foo>abc<bar/>def</foo>”
  ---XTag ”foo”
    |
    +---XText ”abc”
    |
    +---XTag ”bar”
    |
    +---XText ”def”
```

This representation makes the structure obvious, and it is easy to see the relationship to our input string. Let's proceed to extend our XML document with some attributes (taking care to escape the quotes, of course):

```
 Prelude Text.XML.HXT.Parser.XmlParsec> xread ”<foo a1=\”my\” b2=\”oh\”>abc<bar/>def</foo>”
 [NTree (XTag (QN {namePrefix = ””, localPart = ”foo”, namespaceUri = ””})
[NTree (XAttr (QN
 {namePrefix = ””, localPart = ”a1”, namespaceUri = ””})) [NTree (XText ”my”)
[]],NTree (XAttr
 (QN {namePrefix = ””, localPart = ”b2”, namespaceUri = ””})) [NTree (XText
”oh”) []]]) [NTree
 (XText ”abc”) [],NTree (XTag (QN {namePrefix = ””, localPart = ”bar”,
namespaceUri = ””}) [])
 [],NTree (XText ”def”) []]]
```

Notice that attributes are stored as regular NTree nodes with the XAttr content type, and (of course) no children. Feel free to pretty-print this expression, as we did above.

For a trivial example of data extraction, consider this small example using XPath[9]:

```
 Prelude> :set prompt "> "
 > :m + Text.XML.HXT.Parser.XmlParsec Text.XML.HXT.XPath.XPathEval
 > let xml = "<foo>A<c>C</c></foo>"
 > let xmltree = head $ xread xml
 > let result = getXPath "//a" xmltree
 > result
 > [NTree (XTag (QN {namePrefix = "", localPart = "a", namespaceUri = ""}) [])
[NTree (XText "A") []]]
 > :t result
 > result :: NTrees XNode
```

[9] http://en.wikipedia.org/wiki/XPath

89 Using Regular Expressions

Good tutorials where to start

- serpentine.com[1]
- Regular Expressions (Haskell Wiki)[2]

Category:Haskell/Not in book[3]

[1] http://www.serpentine.com/blog/2007/02/27/a-haskell-regular-expression-tutorial/

[2] http://www.haskell.org/haskellwiki/Regular_expressions

[3] https://en.wikibooks.org/wiki/Category%3AHaskell%2FNot%20in%20book

90 Parsing Mathematical Expressions

This chapter discusses how to turn strings of text such as "3*sin x + y" into an abstract syntactic representation like Plus (Times (Number 3) (Apply "sin" (Variable "x"))) (Variable "y").

We are going to use Text.ParserCombinators.ReadP[1] throughout, so you will need to have the reference open to refer to.

90.1 First Warmup

```
import Text.ParserCombinators.ReadP
```

For a warmup, to get started on the problem, we first try an easier problem. A language where the symbols are just the letter "o", a single operator "&" and brackets. First define a data type for these trees:

```
data Tree = Branch Tree Tree | Leaf deriving Show
```

Now a parser for leaves is defined using the ReadP library:

```
leaf = do char 'o'
          return Leaf
```

now to define a parser for the branches, made up by "&" operator we need to choose an associativity. That is, whether o&o&o should be the same as (o&o)&o or o&(o&o) - let us pick the latter.

For a first approximation we can forget about brackets, adding them in after the first "milestone":

```
branch = do a <- leaf
            char '&'
            b <- tree
            return (Branch a b)

tree = leaf +++ branch
```

[1] http://hackage.haskell.org/packages/archive/base/latest/doc/html/Text-ParserCombinators-ReadP.html

It's now possible to test this out and see if it acts properly on a few inputs:

```
   *Main> readP_to_S tree "o"
   [(Leaf,"")]
   *Main> readP_to_S tree "o&o"
   [(Leaf,"&o"),(Branch Leaf Leaf,"")]
   *Main> readP_to_S tree "o&o&o"
   [(Leaf,"&o&o"),(Branch Leaf Leaf,"&o"),(Branch Leaf (Branch Leaf Leaf),"")]
```

Since that worked fine we can proceed to add support for parenthesis. Brackets are defined generally, so that we can reuse it later on

```
   brackets p = do char '('
                   r <- p
                   char ')'
                   return r
```

We can now update the branch and tree parsers to support brackets:

```
   branch = do a <- leaf +++ brackets tree
               char '&'
               b <- tree
               return (Branch a b)

   tree = leaf +++ branch +++ brackets tree
```

A bit of testing shows that it seems to work

```
   *Main> readP_to_S tree "((o&((o&o)))&o&((o&o)&o)&o)"
   [(Branch (Branch Leaf (Branch Leaf Leaf)) (Branch Leaf (Branch (Branch
(Branch Leaf Leaf) Leaf) Leaf)),"")]
```

90.2 Adaptation

This gives a good starting point for adaptation. The first modification towards the ultimate goal, which is quite easy to do, is changing the leaves from just ”o” to any string. To do this we have change to ‘Leaf‘ to ‘Leaf String‘ in the data type and update the leaf function:

```
   data Tree = Branch Tree Tree | Leaf String deriving Show

   leaf = do s <- many1 (choice (map char ['a'..'z']))
             return (Leaf s)
```

For the next adaptation we try and add a new operation ”|” which binders weaker than ”&”. I.e. ”foo&bar|baz” should parse as ”(foo&bar)|baz”. First we need to update the data type representing syntax:

```
data Operator = And | Or deriving Show

data Tree = Branch Operator Tree Tree | Leaf String deriving Show
```

The obvious thing to do is duplicate the ‘branch‘ function and call it ‘andBranch‘ and ‘orBranch‘, and give or precedence using the left choice operator ‘<++‘:

```
andBranch = do a <- leaf +++ brackets tree
               char '&'
               b <- tree
               return (Branch And a b)

orBranch = do a <- leaf +++ brackets tree
              char '|'
              b <- tree
              return (Branch Or a b)

tree = leaf +++ (orBranch <++ andBranch) +++ brackets tree
```

This modification does not work though, if we think of an expression such as ”a&b&c&d|e&f&g&h|i&j&k|l&m&n&o|p&q&r|s” as a tree ”X|Y|Z|W|P|Q” (which we already know how to parse!) except that the leaves are a more complicated form (but again, one we already know how to parse) then we can compose a working parser:

```
andBranch = do a <- leaf +++ brackets tree
               char '&'
               b <- andTree
               return (Branch And a b)

andTree = leaf +++ brackets tree +++ andBranch

orBranch = do a <- andTree +++ brackets tree
              char '|'
              b <- orTree
              return (Branch Or a b)

orTree = andTree +++ brackets tree +++ orBranch

tree = orTree
```

While this approach does work, for example:

```
   *Main> readP_to_S tree "(foo&bar|baz)"
   [(Leaf "","(foo&bar|baz)"),(Branch Or (Branch And (Leaf "foo") (Leaf "bar"))
(Leaf "baz"),""),(Branch Or (Branch And (Leaf "foo") (Leaf "bar")) (Leaf
"baz"),"")]
   *Main> readP_to_S tree "(foo|bar&baz)"
   [(Leaf "","(foo|bar&baz)"),(Branch Or (Leaf "foo") (Branch And (Leaf "bar")
(Leaf "baz")),""),(Branch Or (Leaf "foo") (Branch And (Leaf "bar") (Leaf
"baz")),"")]
```

it parses ambiguously, which is undesirable for efficiency reasons as well as hinting that we may have done something unnatural. Both ‘andTree‘ and ‘orTree‘ functions have ‘brackets tree‘ in them, since ‘orTree‘ contains ‘andTree‘ this is where the ambiguity creeps in. To solve it we simply delete from ‘orTree‘.

```
    orTree = andTree +++ orBranch
```

90.3 Structure Emerges

All the previous fiddling and playing has actually caused a significant portion of the structure of our final program to make its-self clear. Looking back at what was written we could quite easily extend it to add another operator, and another after that (Exercise for the reader: if it is not clear exactly how this would be done, figure it out and do it). A moments meditation suggests that we might complete this pattern and abstract it out, given an arbitrarily long list of operators

```
    operators = [(Or,"|"),(And,"+")]
```

or perhaps

```
    data Operator = Add | Mul | Exp deriving Show

    operators = [(Add,"+"),(Mul,"*"),(Exp,"^")]
```

the parser should be computed from it, nesting it (as we did manually in the past) so that parses happen correctly without ambiguity.

The seasoned haskell programmer will have already seen, in her minds eye, the following:

```
    tree = foldr (\(op,name) p ->
                   let this = p +++ do a <- p +++ brackets tree
                                       char name
                                       b <- this
                                       return (Branch op a b)
                    in this)
                 (leaf +++ brackets tree)
                 operators
```

which is then tested.

```
    *Main> readP_to_S tree "(x^e*y+w^e*z^e)"
    [(Leaf "","(x^e*y+w^e*z^e)"),(Branch Add (Branch Mul (Branch Exp (Leaf "x")
 (Leaf "e")) (Leaf "y")) (Branch Mul (Branch Exp (Leaf "w") (Leaf "e")) (Branch
 Exp (Leaf "z") (Leaf "e"))),"")]
```

This is a good checkpoint to pause, in summary we have distilled the embryonic parser down to the following script:

```
import Text.ParserCombinators.ReadP

brackets p = do char '('
                r <- p
                char ')'
                return r

data Operator = Add | Mul | Exp deriving Show
operators = [(Add,'+'),(Mul,'*'),(Exp,'^')]

data Tree = Branch Operator Tree Tree | Leaf String deriving Show

leaf = do s <- many1 (choice (map char ['a'..'z']))
          return (Leaf s)

tree = foldr (\(op,name) p ->
               let this = p +++ do a <- p +++ brackets tree
                                   char name
                                   b <- this
                                   return (Branch op a b)
                in this)
             (leaf +++ brackets tree)
             operators
```

90.4 Whitespace and applicative notation

Since both the functional/applicative notation and ignoring whitespace depend on some of the same characters (space characters) it is a useful question to ask which should be implemented first, or whether it is not important which should be programmed first.

Considering the expression ”f x”, suggests that we should find how to parse whitespace before handling applicative notation, since once it has been dealt with function application should just correspond to simple juxtaposition (as intended).

There is a technical difficultly making our current parser ignore whitespace: if we were to make a ‘skipWhitespace‘ parser, and put it everywhere that whitespace could occur we would be inundated with ambiguous parses. Hence it is necessary to skip whitespace only in certain crucial places, for example we could pick the convention that whitespace is always skipped *before* reading a token. Then ” a + b * c ” would be seen by the parser chunked in the following way ”[a][+][b][*][c][ ]”. Which convention we choose is arbitrary, but ignoring whitespace before seems slightly neater, since it handles ” a” without any complaints.

We define the following:

```
skipWhitespace = do many (choice (map char [' ','\n']))
                    return ()
```

and update all the parses written before, so that they follow the new convention

```
brackets p = do skipWhitespace
                char '('
                r <- p
```

```
                  skipWhitespace
                  char ')'
                  return r

  leaf = do skipWhitespace
            s <- many1 (choice (map char ['a'..'z']))
            return (Leaf s)

  tree = foldr (\(op,name) p ->
                 let this = p +++ do a <- p +++ brackets tree
                                     skipWhitespace
                                     char name
                                     b <- this
                                     return (Branch op a b)
                  in this)
               (leaf +++ brackets tree)
               operators
```

In order to add applicative support clearly the syntax needs to allow for it:

```
  data Tree = Apply Tree Tree | Branch Operator Tree Tree | Leaf String
deriving Show
```

This syntax tree will allow for sentences such as ”(x + y) foo”, while this not correct other sentences like ”(f . g) x” are commonplace in haskell - it should be the job of the type-checker to decide which is meaningful and which is not: This separation of concerns lets our problem (parsing) remain simple and homogeneous.

Our parser is essentially just two functions ‘leaf‘ and ‘tree‘ (‘skipWhitespace‘ and ‘brackets‘ being considered ”library” or helper functions). The function ‘tree‘ eats up all the operators it can, attaching leaves onto them as it can. While the ‘leaf‘ function could be thought of as reading in anything which doesn’t have operators in it. Given this view of the program it is clear that to support applicative notation one needs to replace leaf with something that parses a chain of functional applications.

The obvious thing to try is then,

```
  leaf = chainl1 (do skipWhitespace
                     s <- many1 (choice (map char ['a'..'z']))
                     return (Leaf s))
                 (return Apply)
```

and it is easily extended to support the ”commonplace” compound sentences discussed earlier:

```
  leaf = chainl1 (brackets tree
                  +++ do skipWhitespace
                         s <- many1 (choice (map char ['a'..'z']))
                         return (Leaf s))
                 (return Apply)
```

This is the problem completely solved! Our original goal is completed, one only needs to specify the operators they would like to have (in order) and write a traversal function converts the ‘Tree‘ into say mathematical expressions -- giving errors if unknown functions were used etc.

90.4.1 Making it Modular

The algorithms written are general enough to be useful in different circumstances, and even if they only had a single use -- if we were planning on using them in a larger program it is essential that we isolate the internals from the externals (its interface).

```
module Parser
 ( Tree(..), parseExpression
 ) where

import Data.Maybe
import Text.ParserCombinators.ReadP

skipWhitespace = do many (choice (map char [' ','\n']))
                    return ()

brackets p = do skipWhitespace
                char '('
                r <- p
                skipWhitespace
                char ')'
                return r

data Tree op = Apply (Tree op) (Tree op) | Branch op (Tree op) (Tree op) | Leaf
String deriving Show

parseExpression operators = listToMaybe . map fst . filter (null .snd) .
readP_to_S tree where
 leaf = chainl1 (brackets tree
                 +++ do skipWhitespace
                        s <- many1 (choice (map char ['a'..'z']))
                        return (Leaf s))
                (return Apply)
 tree = foldr (\(op,name) p ->
                let this = p +++ do a <- p +++ brackets tree
                                    skipWhitespace
                                    char name
                                    b <- this
                                    return (Branch op a b)
                 in this)
              (leaf +++ brackets tree)
              operators
```

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